Algebraic Semantics for Modal Logic with Propositional Quantifiers

Yifeng Ding (voidprove.github.io)

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Department of Philosophy, Peking University

What is modal logic

A logic codifies all the correct ways to use (do reasoning with) certain words.

- Classical propositional logic codifies all the classically correct ways to use 'and', 'or', 'not', which includes distributivity, de Morgan, etc.
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What is modal logic

Modal logics care about a special kind of words called modalities or modal operators, which grammatically are just like ‘and’ and ‘not’: they are applied to sentences to form new sentences.

- ‘There are human residents on the Moon.’ $\mapsto$ ‘It will be the case that there are human residents on the Moon.’
- ‘57 is a prime number.’ $\mapsto$ ‘I believe that 57 is a prime number.’
- ‘There is life in this universe.’ $\mapsto$ ‘It is necessary that there is life in this universe.’

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To justify that certain reasoning patterns are really correct for certain words, we often resort to formal semantics of those words.

Formal semantics typically provides a formal definition of truth condition, and correctness is defined by truth preservation or truth in virtual of meaning.

$$V(\varphi) \in \{0, 1\}$$
$$V(\varphi \vee \psi) = \max(V(\varphi), V(\psi))$$
$$V(\neg\varphi) = 1 - V(\varphi).$$

Then $V(p \vee \neg p) = 1$ no matter whether $V(p) = 0$ or 1.

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Say you want to study an isolated device A evolving in discrete time.

- Let W be the set of all possible states A could be in.
- Let $f : W \to W$ so that $f(x)$ is the next state of A if A is in x.
- Let p be anything meaningful you can say about A.
- Let $V(p) \subseteq W$ be the set of states that render p true.

Then 'It will be that p' is true at precisely those states in $(f^*)^{-1}[V(p)]$.

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Definition

Fix countably infinite set Prop of propositional variables. Then $\mathcal{L}$ is the set of formulas defined by

$$\mathcal{L} \ni \varphi ::= p \mid \top \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \Box\varphi$$

with $p \in \text{Prop}$. $\bot := \neg\top$, $(\varphi \vee \psi) = \neg(\neg\varphi \wedge \neg\psi)$, $(\varphi \rightarrow \psi) := \neg(\varphi \wedge \neg\psi)$, $\Diamond\varphi := \neg\Box\neg\varphi$.

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Definition

A frame is a tuple $F = \langle W, R \rangle$. A valuation V on F is a function from Prop to $\wp(W)$. Then inductively define $\widehat{V} : \mathcal{L} \to \wp(W)$:

$$\widehat{V}(p) = V(p)$$
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$$\widehat{V}(\neg\varphi) = W \setminus \widehat{V}(\varphi)$$
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$$\widehat{V}(\Box\varphi) = \{x \in W \mid R(x) \subseteq \widehat{V}(\varphi)\}.$$

By definition, $\widehat{V}(\Diamond\varphi) = R^{-1}[\widehat{V}(\varphi)]$.

φ is valid on F if for all $V : \text{Prop} \to \wp(W)$, $\widehat{V}(\varphi) = W$.

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Classes $\mathbf{C}$ of frames determine sets of validities $\mathrm{Log}(\mathbf{C}) \subseteq \mathcal{L}$. All of these sets share some properties that bear direct intuitive appeal.

Definition

A set $\mathrm{L} \subseteq \mathcal{L}$ is a normal modal logic if

- L contains all classically valid modality-free formulas
- L is closed under uniform substitution: if $\varphi \in \mathrm{L}$, then $\varphi[\psi/p] \in \mathrm{L}$
- L contains the formula $\mathrm{K} : (\Box p \wedge \Box(p \to q)) \to \Box q$
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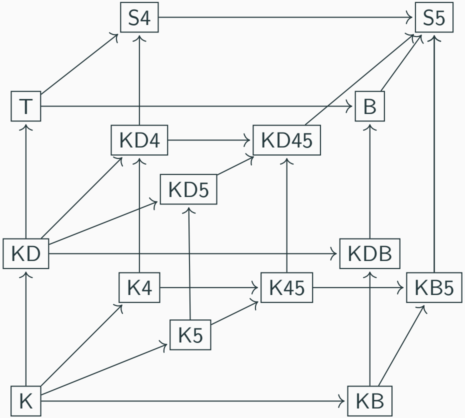
Some normal modal logics



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For the above L, $\mathsf{L} = \mathsf{Log}(\mathsf{Fr}(\mathsf{L}))$.

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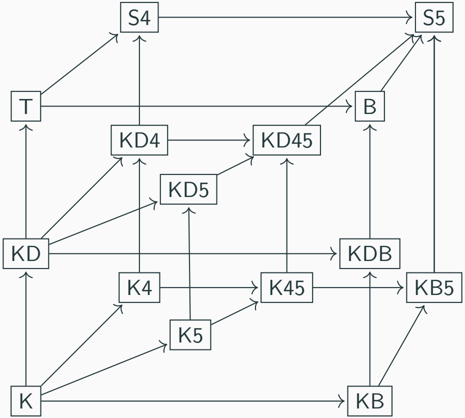

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Examples of propositional quantifiers

$\mathcal{L}$ has no quantifiers, so it's missing quite some expressivity we actually have.

- I believe that everything I believe is true: $B\forall p(Bp \to p)$.
- I know that there's a truth I don't know: $K\exists p(p \land \neg Kp)$.
- a knows that b knows everything a knows: $K_a\forall p(K_a p \to K_b p)$.
- There is a true proposition that necessarily implies every true proposition: $\exists p(p \land \forall q(q \to \Box(p \to q)))$.

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Logical properties of modal logics with propositional quantifiers

- Propositional quantifiers provide extra expressivity.
- The amount of expressivity depends on which modality/which frame class you are talking about.
- Extra expressivity comes with extra complexity in logics, and sometimes the extra complexity can be quite extreme.

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Add propositional quantifiers

Definition

Let $\mathcal{L}\Pi$ be the language with the following grammar

$$\varphi ::= p \mid \top \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \Box\varphi \mid \forall p\varphi$$

where $p \in \text{Prop}$. $\exists p\varphi := \neg\forall p\neg\varphi$.

Let $F = \langle W, R\rangle$ be a frame, and V a valuation on F, then

$$w \in \widehat{V}(\forall p\varphi) \Leftrightarrow \forall V' : \text{Prop} \to \wp(W), \text{ if } V|_{-p} = V'|_{-p} \text{ then } w \in \widehat{V'}(\varphi).$$

Equivalently, $\widehat{V}(\forall p\varphi) = \bigcap_{X\subseteq W} \widehat{V[X/p]}(\varphi)$.

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Examples

Let $F = \langle W, R \rangle$ and V any valuation on F.

- $\widehat{V}(\forall p(\Box p \to p)) = \{w \in W \mid wRw\}$.
- $\widehat{V}(\exists p(p \land \neg\Box p)) = \{w \in W \mid \exists w' \neq w, wRw'\}$.
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In other words, $\exists p(p \land \forall q(q \to \Box(p \to q)))$ is valid on any frame.

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The cost of extra expressivity

You can simulate quantification over binary predicates (over worlds) using propositional quantifiers on an arbitrary Kripke model.

Theorem (Fine 1970)

The logic, with propositional quantifiers, of any of the following classes of frames is recursively equivalent to full second order logic:

- *the class of all frames,*
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More constrained binary relations

As we put more constraints on the binary relation, the simulation of quantification over sets of pairs of worlds becomes harder.

Theorem (Fine 1969)

The logic of the class of equivalence relations is decidable. Over this class of frames, propositional quantifiers essentially add the ability to count worlds.

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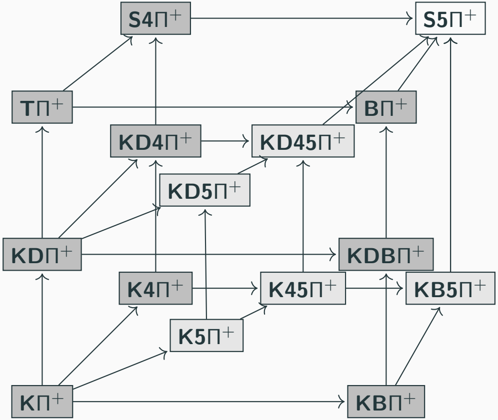
KD45's place in the literature



$\mathsf{L}\Pi^+$ is the set of formulas in $\mathcal{L}\Pi$ valid on all frames validating L.

Expressivity beyond Kripke models

Frames also hide some of the expressivity of propositional quantifiers.

$$\text{At} : \exists p(p \wedge \forall q(q \to \Box(p \to q))),$$
$$\text{Bc} : \forall p \Box\varphi \to \Box\forall p\varphi.$$

At is valid because we have singleton propositions.

Bc is valid because the $\Box$'ed propositions are closed under arbitrary intersections as they are represented by a single set $R(w)$: $w \in \widehat{V}(\Box\varphi)$ iff $\widehat{V}(\varphi) \supseteq R(w)$.

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Algebraic semantics

Propositions form a Boolean algebra when ordered by logical strength. A unary sentential operator is essentially a unary function from propositions to propositions.

Definition

A Boolean algebra expansion (BAE) $\mathcal{B}$ is a tuple $\langle B, \Box \rangle$ where B is a Boolean algebra and $\Box : B \to B$.

For any V : Prop $\to B$, $\widehat{V}(\Box\varphi) = \Box(\widehat{V}(\varphi))$.

For any $b \in B$, $\widehat{V}(\forall p \varphi) \leqslant \widehat{V[b/p]}(\varphi)$.

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A BAE $\mathcal{B} = \langle B, \Box \rangle$ is complete if any $X \subseteq B$ has a highest lower bound (meet) in B.

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This generalizes the frame-based semantics for $\forall p$ by replacing $\bigcap$ with $\bigwedge$.

A formula φ is valid on $\langle B, \Box \rangle$ if for all $V : \mathrm{Prop} \to B$, $\widehat{V}(\varphi) = \top$.

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Definition

The following are called Π-principles.

- Dist : $\forall p(\varphi \to \psi) \to (\forall p\varphi \to \forall p\psi)$,
- Inst : $\forall p\varphi \to \varphi[\psi/p]$, if substitutable,
- Vacu : $\varphi \to \forall p\varphi$, if p is not free in φ,
- Univ : whenever φ is a logic, $\forall p\varphi$ is also the logic.

$\mathsf{L} \subseteq \mathcal{L}\Pi$ is a normal Π-logic if it satisfies the Π-principles and also the requirements for being a normal modal logic in $\mathcal{L}\Pi$.

If L is a normal modal logic, LΠ is the smallest normal Π-logic extending L.

Logics extending S5

S5 is the smallest normal modal logic containing

$$T : \Box p \to p \quad 4 : \Box p \to \Box\Box p \quad 5 : \neg\Box p \to \Box\neg\Box p.$$

S5 is also the logic of the class **U** of all frames $\langle W, W \times W \rangle$.

Theorem (Scrogg)

Every normal modal logic extending S5 is the logic of a subset of **U**.

These logics form a chain isomorphic to reversed $\mathbb{N}^{\infty}$ and are all computable.

But S5Π is not the logic of any class of frames, because At $\notin$ S5Π.

Complete simple monadic algebras

Definition

A complete simple monadic algebra is a complete BAE $\langle B, \Box \rangle$ such that

$$\Box a = \begin{cases} \top & \text{if } a = \top \\ \bot & \text{otherwise.} \end{cases}$$

These are "simple" complete BAEs validating S5. They generalize frames with a universal relation.

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Logics extending $S5\Pi$

Theorem

Every normal Π*-logic containing* S5 *is the logic of the class of complete simple monadic algebras validating it.*

There are a continuum many such logics, ordered by inclusion like open sets in the disjoint union of two copies of $\mathbb{N}^\infty$.

For every $X \subseteq \mathbb{N}$*, there is such a logic Turing-equivalent to* X.

A pair $(n, t) \in \mathbb{N}^\infty \times \{0, 1\}$ represents the type of a complete Boolean algebra: n is the number of atoms it has, and $t = 1$ iff it has an atomless part.

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Logics of belief with propositional quantifiers

Without propositional quantifiers, KD45 is a natural logic for belief, and is known for decades.

K	$(\Box p \wedge \Box(p \to q)) \to \Box q$	D	$\neg\Box(p \wedge \neg p)$
4	$\Box p \to \Box\Box p$	5	$\neg\Box p \to \Box\neg\Box p$
Nec	$\vdash \varphi \Rightarrow \vdash \Box\varphi$.		

A KD45 frame is a frame with a serial, transitive, and Euclidean relation: $(xRy \wedge xRz) \Rightarrow yRz$.

$R(w)$ is the set of *uneliminated possibilities* you have at w. $R(w) \neq \varnothing$, and for any $w' \in R(w)$, $R(w') = R(w)$ b/c you know what $R(w)$ is.

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Facing choices

With propositional quantifiers, we will need to decide whether the following should sit in a *logic* of belief:

- At: $\exists p(p \to \forall q(q \to \Box(p \to q)))$.
- Bc: $\forall p \Box \varphi \to \Box \forall p \varphi$.
- Immod: $\Box \forall p(\Box p \to p)$.
- $4^{\forall}$: $\forall p \Box \varphi \to \Box \forall p \Box \varphi$.

$4^{\forall}$ is still about introspection. But the others require more.

Immod doesn't sound right.

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Algebraic semantics

KD45 frames validate At, Bc, Immod. We need to generalize.

Definition

A KD45 algebra is a BAE validating KD45. These are also called pseudo-monadic algebras.

A well-connected KD45 algebra (wKD45 algebra) is a BAE $\langle B, \Box \rangle$ such that there is a proper filter $F_{\mathcal{B}}$ in B such that

$$\Box a = \begin{cases} \top & \text{if } a \in F_{\mathcal{B}} \\ \bot & \text{otherwise.} \end{cases}$$

These are KD45 algebras $\langle B, \Box \rangle$ such that $range(\Box) = \{\top, \bot\}$.

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Main results

Algebra class	Logic
complete wKD45 algebras	$\mathsf{KD4}^{\forall}\mathsf{5}\Pi$
complete wKD45 algebras with $F_{\mathcal{B}}$ principal	$\mathsf{KD4}^{\forall}\mathsf{5}\Pi\mathsf{Immod}$
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KD45 Kripke models	$\mathsf{KD4}^{\forall}\mathsf{5}\Pi\mathsf{AtImmod}$

All of these are decidable. More generally

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The logic of complete wKD45 algebras validating φ is $\mathsf{KD4}^{\forall}\mathsf{5}\Pi\varphi$. The logic of atomic complete wKD45 algebras validating φ is $\mathsf{KD4}^{\forall}\mathsf{5}\Pi\mathsf{At}\varphi$.

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Any complete BAE validating KD45 validates $4^\forall$*.*

If the structure of propositions is complete, then $4^\forall$ is a logical consequence of KD45.

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$4^\forall$ *is not in* $\mathrm{KD4}^\forall 5\Pi\mathrm{Immod}$*. Thus there is a BAE validating KD45 but not* $4^\forall$*.*

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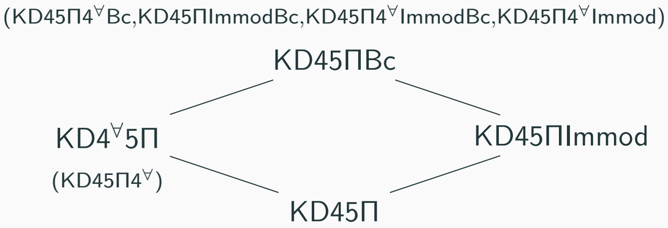
Even with the help of Immod, Π-principles can't derive $4^\forall$ from KD45. The refutation uses a model with propositional contingency where the agent modeled thought there are more propositions than there really is.



(KD45Π4^∀Bc,KD45ΠImmodBc,KD45Π4^∀ImmodBc,KD45Π4^∀Immod)
KD45ΠBc
KD4^∀5Π
(KD45Π4^∀)
KD45ΠImmod
KD45Π

Proof strategy

A big part of the language is really talking about the quotient Boolean algebra $\mathcal{B}/F_{\mathcal{B}}$ in a first order way.

For example $\widehat{V}(\Diamond p)$ is either $\top$ or $\bot$ and is $\top$ iff $\widehat{V}(p) \neq \bot$ in $\mathcal{B}/F_{\mathcal{B}}$. Then

$$\widehat{V}(\forall p(\Diamond p \to \exists q(\Diamond(p \wedge q) \wedge \Diamond(p \wedge \neg q))))$$

is either $\top$ or $\bot$, and it is $\top$ iff $\mathcal{B}/F_{\mathcal{B}}$ is atomless: for all $b \in \mathcal{B}/F_{\mathcal{B}}$, if $b > \bot$ then there is $\bot < b' < b$ in $\mathcal{B}/F_{\mathcal{B}}$.

Proof strategy

But not all formulas are evaluated to either $\top$ or $\bot$. $\widehat{V}(\forall p(\Box p \to p))$ is always $\bigwedge F_{\mathcal{B}}$ which typically is neither $\top$ nor $\bot$.

We can separate the translatable part from the non-translatable part.

After translation, we just need:

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The first-order logic of the non-trivial quotients of complete Boolean algebras is just the first-order logic of all non-trivial Boolean algebras.

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Theorem (Vermeer 1996)

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Two Boolean algebras are elementarily equivalent iff they have the same invariant.

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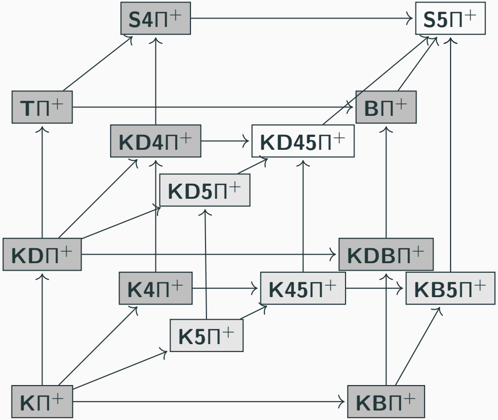
Reviewing the results



Here $L\Pi^+$ is the Π-logic of all frames validating L.

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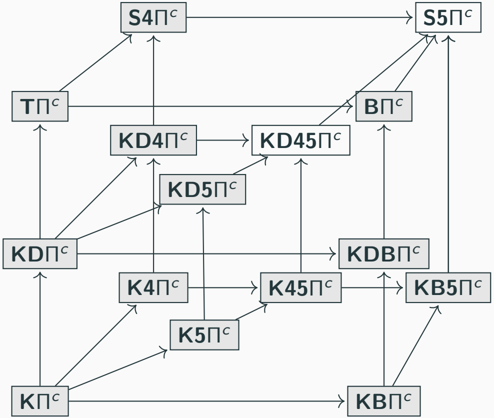

Here LΠ^c is the Π-logic of all complete BAEs validating L.

Puzzles remain

$\mathsf{S5\Pi}^c = \mathsf{S5\Pi}$, the logic of all complete simple monadic algebras.

However, $\mathsf{KD45\Pi}^c \neq \mathsf{KD45\Pi}$ because of $4^\forall$.

What are some intrinsic sufficient/necessary condition for $\mathsf{L\Pi}^c = \mathsf{L\Pi}$?

We know that $\mathsf{S4\Pi}^+$ is highly non-c.e. and $\mathsf{S4\Pi}$ is trivially c.e., but what about $\mathsf{S4\Pi}^c$?

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Thank you!