On the Logics with Propositional Quantifiers Extending S5Π

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Introduction

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- Kit Fine systematically studied a few modal logic systems with propositional quantifers based on S5.
- We provide an analogue of Scroggs's theorem for modal logics with propositional quantifiers using algebraic semantics.
- More generally, it is interesting to see how classical results generalize when using algebraic semantics.

Outline

Review of Kripke Semantics

Algebraic Semantics

Main Theorems

Future Research

Review of Kripke Semantics

Language

Definition

Let $\mathcal{L}\Pi$ be the language with the following grammar

$$\varphi ::= p \mid \top \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \Box\varphi \mid \forall p\varphi$$

where $p \in \text{Prop}$, a countably infinite set of propositional *variables*. Other Boolean connectives, $\bot$, and $\Diamond$ are defined as usual.

Kripke semantics

Every subset is a proposition!

- A pointed model $\langle W, R, V\rangle, w$ makes $\forall p\varphi$ true iff for all $X \subseteq W$, $\langle W, R, V[p \mapsto X]\rangle, w$ makes φ true.
- Equivalently, $[\![\forall p\varphi]\!]^{\mathcal{M}} = \bigcap_{X\subseteq\mathcal{M}} [\![\varphi]\!]^{\mathcal{M}[p\mapsto X]}$.

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$[\![\forall p (\Box \Diamond p \to \Diamond \Box p)]\!]^{\mathcal{M}}$ is not first-order definable.

Kripke semantics

Another example:

$$[\![\Diamond p \wedge \forall q(\Box(p \to q) \vee \Box(p \to \neg q))]\!]^{\mathcal{M}}$$

is the set of points that can access to exactly one element in $V(p)$.

Call this formula *atom*(p).

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Theorem

Full second-order logic can be embedded into SOPML (preserving satisfiability) when R is S4.2 or weaker.

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Theorem

When $R = W \times W$, SOPML is expressively equivalent to MSO.

Algebraic Semantics

Algebraic semantics: reasons

- Kripke frames corresponds to complete, atomic, completely multiplicative modal algebras. We are forced to accept $\exists p(p \wedge atom(p))$ when $\Box$ is S5. And we are forced to accept Barcan: $\forall p \Box \varphi \leftrightarrow \Box \forall p \varphi$.

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General Π logics

Definition

A (normal) Π-logic is a set Λ of formulas in $\mathcal{L}\Pi$ such that it is first of all a (normal modal logic) propositional modal logic and that it contains

- $\forall p(\varphi \to \psi) \to (\forall p\varphi \to \forall p\psi)$
- $\forall p\varphi(p) \to \varphi(\psi)$
- $\varphi \to \forall p\varphi$ when p is not free

and is closed under universalization: $\varphi/\forall p\varphi$.

The smallest normal Π-logic containing a normal modal logic L is called LΠ.

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Of course this is because the atomicity.

General algerbaic semantics gives precisely S5Π.

Algebraic semantics

Definition

For any modal algebra B, a *valuation* V on B is a function from Prop to B. It naturally extends to $\widehat{V} : \mathcal{L} \to B$ in the usual way.

When B is complete, any such valuation can then be extended to an $\mathcal{L}\Pi$-*valuation* $\widehat{V} : \mathcal{L}\Pi \to B$ by setting

- $\widehat{V}(\forall p\varphi) = \bigwedge\{\widehat{V[p \mapsto b]}(\varphi) \mid b \in B\}$.

A formula $\phi \in \mathcal{L}\Pi$ is *valid* on a complete modal algebra B, written as $B \vDash \phi$, if for all valuations V on B, $\widehat{V}(\phi) = 1$.

Galois connection

A simple Galois connection:

$$\mathrm{Log}(\mathcal{C}) = \{\varphi \in \mathcal{L}\Pi \mid B \vDash \varphi \text{ for all } B \in \mathcal{C}\}$$
$$\mathrm{Alg}(X) = \{B \text{ a complete modal algebra} \mid B \vDash X\}$$

For any class $\mathcal{C}$ of complete modal algebras, $\mathrm{Log}(\mathcal{C})$ is a normal Π-logic.

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Questions

Which normal Π-logics are complete? Characterize those Λ such that $\Lambda = \mathrm{Log}(\mathrm{Alg}(\Lambda))$.

Which classes of complete modal algebras are variety-like? Charaterize those $\mathcal{C}$ such that $\mathrm{Alg}(\mathrm{Log}(\mathcal{C}))$.

Simple S5 algebras

A simple S5 algebra is a Boolean algebra together with an propositional discriminator $\Box$:

$$\Box\top = \top; \Box b = \bot \text{ for all } b \neq \top.$$

Call them csS5A. Then we have the completeness of S5Π.

$$\mathrm{Log}(\mathrm{csS5A}) = \mathrm{S5\Pi}.$$

Main Theorems

General completeness

Theorem

For all normal Π*-logic* $\Lambda \supseteq \mathsf{S5}\Pi$,

$$\mathsf{Log}(\mathsf{Alg}(\Lambda) \cap \mathsf{csS5A}) = \Lambda.$$

Note that this is different than: for all $\mathsf{L}\Pi$ where L is a modal logic extending S5, it is complete (w.r.t. its csS5As).

Lattice structure

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Theorem

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What it is really like:

$$\begin{array}{cccccccc} & 1 & 2 & 3 & 4 & \cdots & \infty \\ 0' & 1' & 2' & 3' & 4' & \cdots & \infty' \end{array}$$

Non-normal Π-logics above S5Π

$S5\Pi + \exists p(p \wedge atom(p))$ is non-normal.

The logic is given by the class of simple complete S5 algebras with the filter of atomic elements as the designated set of "truth values".

Proof idea: expressivity

The idea of the proof: we can calculate the expressivity of $\langle \mathcal{L}\Pi, \text{csS5A}, \vDash \rangle$, and the expressvity is reflected syntactically in S5Π.

Then we can determine the classes of csS5As that are characterized by logics.

Expressivity

Definition

Let g be $\exists p(p \wedge atom(p))$. Let $\mathrm{M}_i\varphi$ be

$$\exists q_1 \cdots \exists q_n(\bigwedge_{1 \leqslant i < j \leqslant n} \Box(q_i \to \neg q_j) \wedge \bigwedge_{1 \leqslant i \leqslant n} (\mathrm{atom}(q_i) \wedge \Box(q_i \to \varphi)))$$

Let $\mathcal{S}$Basic be the following fragment of $\mathcal{L}\Pi$:

$$\varphi ::= \top \mid \Diamond\neg \mathrm{g} \mid \mathrm{M}_i\top \mid \neg\varphi \mid (\varphi \wedge \varphi).$$

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Theorem

There is a function basic : $\mathcal{L}\Pi \to \mathcal{S}$Basic *such that* $B \vDash \varphi$ *iff* $B \vDash basic(\varphi)$ *and* $\mathsf{S5}\Pi \vdash \Box u(\varphi) \leftrightarrow basic(\varphi)$.

Tarski invariant

$\Diamond\neg g$ says "there is an atomless proposition". $\mathsf{M}_i\top$ says "there are at least i many atoms".

Definition

For any csS5A B, its *type* $t(B)$ is a pair $\langle t_0(B), t_1(B)\rangle$ where

$$t_0(B) = \begin{cases} 1 & \text{if } B \text{ is atomic} \\ 0 & \text{if } B \text{ is not atomic,} \end{cases}$$

$$t_1(B) = \begin{cases} i \in \mathbb{N} & \text{if } B \text{ has exactly } i \text{ atoms} \\ \infty & \text{if } B \text{ has infinitely many atoms.} \end{cases}$$

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Theorem

$B \equiv_{\mathcal{L}\Pi} B'$ iff $t(B) = t(B')$.

Type space

The types are:

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And $\mathcal{S}\text{Basic} \ni \varphi ::= \top \mid \Diamond\neg \mathsf{g} \mid \mathsf{M}_i\top \mid \neg\varphi \mid (\varphi \wedge \varphi)$ makes this set a Stone space.

Theorem

Let $\text{Type}(\varphi) = \{t(B) \mid a\ csS5A\ B \vDash \varphi\}$. *Then the type space* $\langle t(csS5A), \text{Type}(\mathcal{S}\text{Basic})\rangle$ *is a Stone space.*

Type space

Observations:

- The type space is also the Stone space of the Lindenbaum algebra of the propositional logic in $\mathcal{S}$Basic with axioms $\mathsf{M}_{i+1}\top \to \mathsf{M}_i\top$ and $\neg\mathsf{M}_0\top \to \Diamond\neg g$.

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- For any Λ a normal Π-logics above S5Π, Type(Λ) is a filter of basic clopens. $\mathrm{Log}(\bigcap \mathrm{Type}(\Lambda)) = \Lambda$ by compactness. Hences logics and closed sets are in one-to-one correspondence.

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- In fact, the normal Π-logics extending S5Π are theories of S5Π. This can be seen by first restricting them to $\mathcal{S}$Basic.

Future Research

Completeness questions

Questions

Which Π-logics are complete? Characterize those Λ such that $\Lambda = \mathsf{Log}(\mathsf{Alg}(\Lambda))$.

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Also:

Question

For which modal logic L that is complete w.r.t. complete modal algebras is LΠ also complete w.r.t. complete modal algebras?

Conservativity questions

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Is there a $\mathcal{C}$-incomplete normal modal logic L which still has $\mathsf{L} = \mathsf{L}\Pi \cap \mathcal{L}$?

Soundness question

For any normal Π-logic, we can still construct its Lindenbaum algebra, which is in general not complete, but the required meets are there for the semantics to be well defined.

Question

For a given Π-logic, find meaningful characterizations of the modal algebrase on which the semantics is always well-defined.

In particular, when is this going to be a first-order condition?