The Logic of Comparative Cardinality

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Introduction

A field of sets

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A field of sets $(X, \mathcal{F})$ is a pair where

1. X is a set and $\mathcal{F} \subseteq \wp(X)$;
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- But more information can be extracted from a field of sets.
- We compare their sizes.

Comparing the sizes

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Given a countably infinite set Φ of set labels, the language $\mathcal{L}$ is generated by the following grammar:

$$\begin{array}{rcl} t & ::= & a \mid t^c \mid (t \cap t) \\ \varphi & ::= & |t| \geq |t| \mid \neg\varphi \mid (\varphi \wedge \varphi), \end{array}$$

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- Terms are evaluated by $\widehat{V}$ on $\mathcal{F}$ in the obvious way.
- $|s| \geq |t|$: set s is at least as large as set t: there is an injection from $\widehat{V}(t)$ to $\widehat{V}(s)$.

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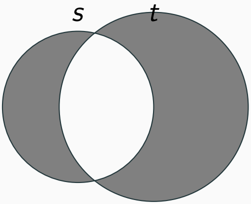
- 

- For finite sets s, t, $|s| \geq |t| \leftrightarrow |s \cap t^c| \geq |t \cap s^c|$.
- For infinite sets s, t, u
    - $|s| \geq |t| \rightarrow |s \cap t^c| \geq |t \cap s^c|$ is not valid;
    - $(|s| \geq |t| \wedge |s| \geq |u|) \rightarrow |s| \geq |t \cup u|$ is valid.

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- The sentences in $\mathcal{L}$ valid on infinite sets have been axiomatized, with size interpreted as likelihood or possibilities.
- We want to combine them: with no extra constraint on $(X, \mathcal{F})$, what is the logic?

Outline

Laws common to finite and infinite sets

Definition (BasicCompLogic)

Boolean reasoning on the sentence level.

Boolean Reasoning on the set level.

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$\geq$ works well with $\varnothing$.

- $\neg\, |\varnothing| \geq |\varnothing^c|$;
- $(|\varnothing| \geq |s| \wedge |\varnothing| \geq |t|) \rightarrow |\varnothing| \geq |s \cup t|$;

From logic to algebra

Definition

A *comparison algebra* is a pair $\langle B, \succeq \rangle$ where B is a Boolean algebra and $\succeq$ is a total preorder on B such that

- for all $a, b \in B$, $a \geq_B b$ implies $a \succeq b$,
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Any formula φ consistent with BasicCompLogic is satisfiable in a finite comparison algebra, with $\succeq$ interpreting $|\cdot| \geq |\cdot|$.

$$\varphi \Rightarrow \Sigma \Rightarrow \langle B, \succeq, V \rangle \Rightarrow \langle \quad \underbrace{V(T(var(\varphi)))}_{\text{relevant terms, a finite set}} \quad , \succeq, V \rangle$$

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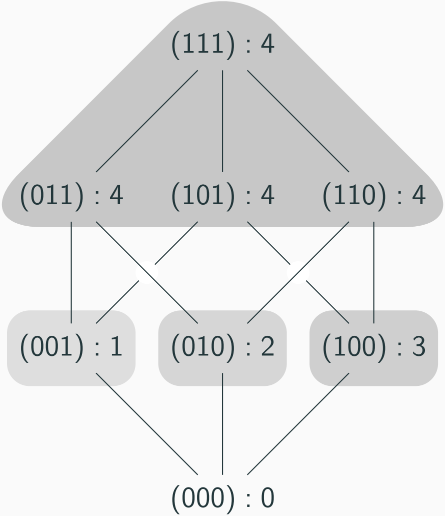
$$\not\Rightarrow \langle X, \mathcal{F}, V \rangle$$

Not enough constraints



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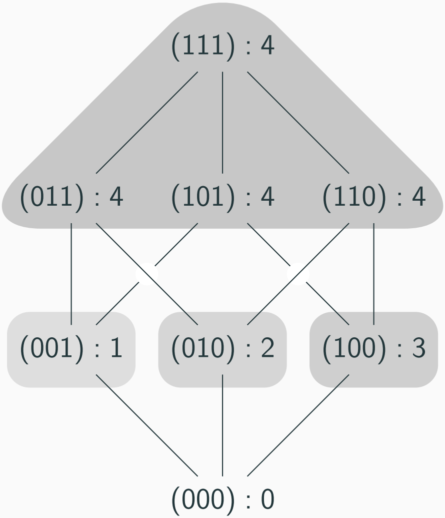

- (010) and (100) should be finite.
- Then all must be finite.
- But $|(011)| = |(101)|$ while $|(010)| < |(100)|$.

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Message

- Any formula φ consistent with BasicCompLogic is satisfiable in a finite comparison algebra $\mathcal{B}$, with $\succeq$ interpreting $|\cdot| \geq |\cdot|$.
- But the ordering $\succeq$ in this $\mathcal{B}$ might not be based on any cardinality comparison.
- We need to know when the ordering arise from cardinality comparison, and add the constraints to the logic.

Plan

Definitions

Definition

A *measure algebra* is a pair $\langle B, \mu \rangle$, where B is a Boolean algebra and μ is a function assigning a cardinal to each element of B such that

- if $a \wedge b = \bot$, then $\mu(a \vee b) = \mu(a) + \mu(b)$, and
- $\mu(b) = 0$ iff $b = \bot$.

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A comparison algebra $\langle B, \succeq \rangle$ is *represented by* a measure algebra $\langle B, \mu \rangle$ if for all $a, b \in B$, we have $a \succeq b$ iff $\mu(a) \geq \mu(b)$.

A representation theorem for finite sets

Theorem (Kraft, Pratt, Seidenberg)

For any finite comparison algebra $\langle B, \succeq \rangle$, it is represented by a measure algebra $\langle B, \mu \rangle$ where Range$(\mu) = \omega$ if and only if:

- *for any two sequences of elements $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ from B, if every atom of B is below (in the order of the Boolean algebra) exactly as many a's as b's, and if $a_i \succeq b_i$ for all $i \in \{1, \ldots, n-1\}$, then $b_n \succeq a_n$.*

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We call this condition "finite cancellation"

Finite cancellation illustrated

$$\begin{matrix} a_0 & + & a_1 & + & a_2 & & = & & b_0 & + & b_1 & + & b_2 \\ \begin{bmatrix}1\\0\\1\\0\end{bmatrix} & + & \begin{bmatrix}1\\1\\1\\1\end{bmatrix} & + & \begin{bmatrix}0\\0\\1\\1\end{bmatrix} & = & \begin{bmatrix}2\\1\\3\\2\end{bmatrix} & = & \begin{bmatrix}0\\1\\1\\1\end{bmatrix} & + & \begin{bmatrix}1\\0\\1\\1\end{bmatrix} & + & \begin{bmatrix}1\\0\\1\\0\end{bmatrix} \end{matrix}$$

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- Then $|a_0| + |a_1| + |a_2| = |b_0| + |b_1| + |b_2|$.
- Then if $|a_0| \geq |b_0|$ and $|a_1| \geq |b_1|$, we can't have $|a_2| > |b_2|$, which means $|b_2| \geq |a_2|$.

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Then $\mathcal{B}$ is represented by a finite measure algebra $m(\mathcal{B}) = \langle B, \mu \rangle$ such that $a \in F$ iff $\mu(a)$ is finite.

Plan

The logic with extra predicates (With FC)

Add the following to BasicCompLogic, $\text{CardCompLogic}_{\mathrm{Fin,Inf}}$ is sound and complete w.r.t. measure algebras and fields of sets.

1. $\mathrm{Fin}(s) \oplus \mathrm{Inf}(s)$;
2. $\bigwedge_i \mathrm{Fin}(s_i) \to \mathrm{Fin}(\bigcup_i s_i)$; $(\mathrm{Fin}(t) \wedge s \subseteq t) \to \mathrm{Fin}(s)$;
3. $(\mathrm{Fin}(s) \wedge \mathrm{Inf}(t)) \to |t| > |s|$;
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5. $\bigwedge_{i=1}^{n}(\mathrm{Fin}(s_i) \wedge \mathrm{Fin}(t_i)) \to \mathsf{FC}(s_1, \cdots, s_n, t_1, \cdots, t_n)$,

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But that's assuming that we can distinguish finite and infinite sets, which uses two extra predicates.

Plan

Defining finiteness and infiniteness

Can we define Fin and Inf in the language of pure cardinality comparison?

Defining finiteness and infiniteness

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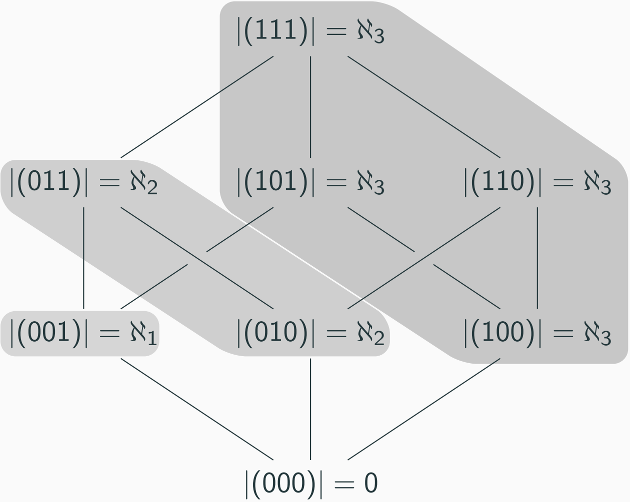
No. There are models that satisfy exactly the same formulas in $\mathcal{L}$, but one has only infinite sets and the other has a finite set.

Undefinability



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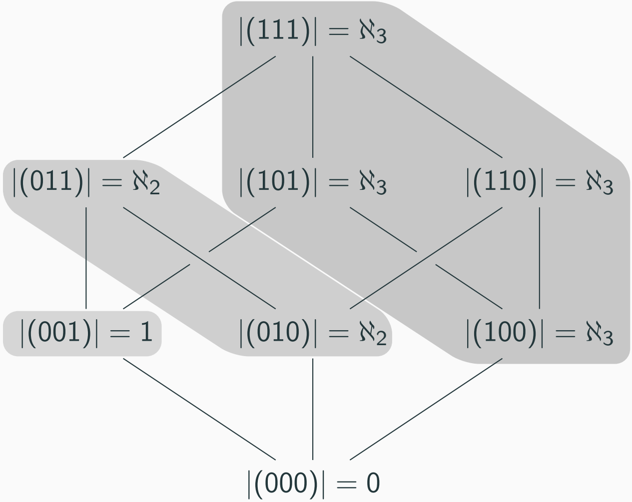

Flexible algebras

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For any flexible measure algebra $\langle B, \mu \rangle$ and any cardinal κ, there exists a flexible measure algebra $\langle B, \mu' \rangle$ such that

- μ(the smallest atom) $= \kappa$;
- For any $a, b \in B$, $\mu(a) \geq \mu(b)$ iff $\mu'(a) \geq \mu'(b)$;
- $\langle B, \mu, V \rangle \equiv_{\mathcal{L}} \langle B, \mu', V \rangle$ for any valuation V.

Defining Fin and Inf

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We must do our best. Perhaps flexible models are the only models where finiteness can't be defined?

Defining Fin and Inf

When $\Delta \subseteq \Phi$ is finite, define $\mathrm{Fin}_\Delta(u)$ for any set term $u \in T(\Delta)$ as:

$$\bigvee_{\substack{R \subseteq T_0(\Delta) \\ S,T \in T_0(\Delta)^{|R|}}} \begin{cases} u = \bigcup_{i=1}^{|R|} r_i \\ \bigwedge_{i=1}^{|R|} \begin{cases} |s_i \cup t_i| > |s_i| \geq |t_i| \\ |s_i \cup t_i| \geq |r_i| \end{cases} \end{cases}$$

Here r_i ranges over elements in R, and s_i, t_i range over the elements in sequences S and T, respectively.

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u is the union of r_i's

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Defining Fin and Inf

When $\Delta \subseteq \Phi$ is finite, define $\mathrm{Fin}_\Delta(u)$ for any set term $u \in T(\Delta)$ as:

$$\bigvee_{\substack{R \subseteq T_0(\Delta) \\ S,T \in T_0(\Delta)^{|R|}}} \begin{cases} u = \bigcup_{i=1}^{|R|} r_i \\ \bigwedge_{i=1}^{|R|} \begin{cases} |s_i \cup t_i| > |s_i| \geq |t_i| \\ |s_i \cup t_i| \geq |r_i| \end{cases} \end{cases}$$

Here r_i ranges over elements in R, and s_i, t_i range over the elements in sequences S and T, respectively.

u is the union of r_i's

There exists r_i, s_i, t_i's

s_i, t_i and also r_i's are finite

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When $\Delta \subseteq \Phi$ is finite, define $\mathsf{Inf}_\Delta(u) :=$ for any set term $u \in T(\Delta)$ as:

$$\bigvee_{s,t \in T_0(\Delta)} \left(t \not\subseteq s \ \wedge \ |u| \geq |s| \geq |s \cup t|\right)$$

Definition works

For any measure algebra model $\langle B, \mu, V \rangle$ such that every element is named by a term in $T(\Delta)$, namely $V(T(\Delta)) = B$:

- If $\mathrm{Fin}_\Delta(u)$ is true, then $\mu(\widehat{V}(u))$ is finite.
- If $\mathrm{Inf}_\Delta(u)$ is true, then $\mu(\widehat{V}(u))$ is infinite.
- $\mathrm{Fin}_\Delta(u)$ and $\mathrm{Inf}_\Delta(u)$ can't be both true.
- If they are both false, then $\langle B, \mu \rangle$ is flexible, and $\widehat{V}(u)$ is the smallest atom.
- $(s \subseteq t \wedge \mathrm{Fin}(t)) \rightarrow \mathrm{Fin}(s)$ and $(\mathrm{Fin}(s) \wedge \mathrm{Fin}(t)) \rightarrow \mathrm{Fin}(s \cup t)$ are derivable in BasicCompLogic.

The axioms

Definition

Where $\Delta \subseteq \Phi$ is finite, define $\mathrm{Axiom}(\Delta)$ as the set containing all of the following formulas for all $u, s, t \in T_0(\Delta)$:

1. $\neg(\mathrm{Fin}_\Delta(u) \wedge \mathrm{Inf}_\Delta(u))$;
2. $(\neg\mathrm{Fin}_\Delta(u) \wedge \neg\mathrm{Inf}_\Delta(u)) \rightarrow$
   $$\bigwedge_{t \in T_0(\Delta)}(|u| \geq |t| \rightarrow (t = \varnothing \vee t = u));$$
3. $(\mathrm{Fin}_\Delta(s) \wedge \mathrm{Fin}_\Delta(t)) \rightarrow (|s| \geq |t| \leftrightarrow |s \cap t^c| \geq |t \cap s^c|)$;
4. $\mathrm{Inf}_\Delta(u) \rightarrow ((|u| \geq |s| \wedge |u| \geq |t|) \rightarrow |u| \geq |s \cup t|)$;
5. $(\mathrm{Inf}_\Delta(s) \wedge \mathrm{Fin}_\Delta(t)) \rightarrow |s| > |t|$.

The logic

Definition

Let CardCompLogic be the logic for $\mathcal{L}$ with the following axioms and rules:

1. all axioms and rules in BasicCompLogic;
2. for any finite $\Delta \subseteq \Phi$, all formulas in Axioms(Δ);
3. the polarizability rule (A7).

Proof sketch

Pick a φ consistent with CardCompLogic, take $\Delta = var(\varphi)$:

1. Extend it to Σ maximally consistent in CardCompLogic.
2. Σ is also maximally consistent with BasicCompLogic. Get canonical comparison model $\mathcal{C}$.
3. Restrict $\mathcal{C}$ to terms in $T(\Delta)$, get $\mathcal{B}$.
4. $\mathcal{B} \models \text{Axiom}(\Delta)$ and also $\text{Fin}(\vec{s}) \rightarrow \text{FC}(\vec{s})$. Use the terms with Fin as finite elements. Apply representation theorem and get measure algebra model $\mathcal{M} \equiv_{\mathcal{L}} \mathcal{B}$.
5. $\mathcal{M} \models \varphi$. So φ is satisfiable.

Conclusion

The logic of cardinal comparison on arbitrary fields of sets can be axiomatized by putting together

- a basic system for orderings extending the inclusion ordering;
- a working definition for finiteness and infiniteness based on witnesses;
- characteristic axioms and rules for finite and infinite sets.

The axiomatization is weak; the logic is non-compact. We use finite Boolean algebras in an essential way.

Plan

Further questions

Representation theorems in the infinite:

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- A field of sets $\langle X, \mathcal{F} \rangle$ ($\mathcal{F}$ possibly infinite) naturally give rise to a measure algebra $\langle \mathcal{B}, \mu \rangle$. The $\mathcal{B}$ part can be arbitrary due to Stone duality. But what about μ?

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Our logic is not strongly complete, as it is finitary but not compact. What is the strongly complete logic?

Non-compactness

The problem of finiteness:

$$\{|s_n| < |s_{n+1}| \mid n \in \omega\} \cup \{|s_n| \leq |t| \mid n \in \omega\} \cup \{\mathrm{Fin}(t)\}.$$

The problem of well-foundedness:

$$\{|s_{n+1}| < |s_n| \mid n \in \omega\}.$$

The problem of discreteness:

$$\text{Disjoint}\{t_i, s_i \mid i \in \omega\} \cup$$
$$\{|t_i| = |t_j|, |s_i| = |s_j| \mid i, j \in \omega\} \cup$$
$$\{| \cup_{i<m_1} t_i| < | \cup_{i<n} s_i| < | \cup_{i<m_2} t_i| \mid (\frac{m_1}{n}, \frac{m_2}{n}) \overset{\lim}{\to} \sqrt{2}\}.$$

Thank You.

Definition of FC

Definition

For each sequence of n terms $\vec{s} = \langle s_0, \cdots, s_{n-1} \rangle$ and $f \in {}^n2$, define

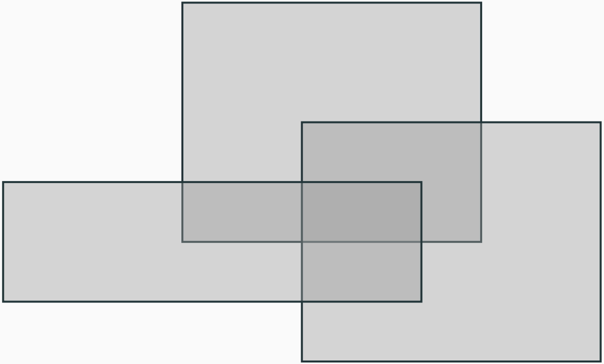
$$\vec{s}[f] = \bigcap\{s_i \mid f(i) = 1\} \cap \bigcap\{s_i^c \mid f(i) = 0\},$$
$$\mathrm{N}_m(\vec{s}) = \bigcup\{\vec{s}[f] \mid f : n \to 2 \text{ and } |f^{-1}(1)| = m\}.$$

Given two sequences $\vec{s}$ and $\vec{t}$ of n terms, define

$$\vec{s}\ \mathrm{E}\ \vec{t} = \bigwedge_{0 \leq i \leq n} (\mathrm{N}_i(\vec{s}) = \mathrm{N}_i(\vec{t})),$$
$$\mathrm{FC}(\vec{s}, \vec{t}) = \vec{s}\ \mathrm{E}\ \vec{t} \to ((\bigwedge_{i < n-1} |s_i| \geq |t_i|) \to |t_{n-1}| \geq |s_{n-1}|).$$

Polarization

With polarization, we can almost do set addition:



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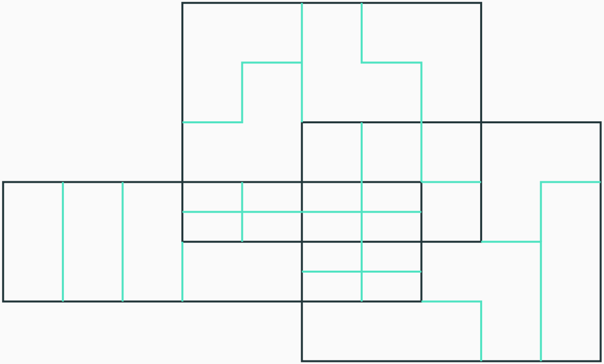

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