Modal Logics with Non-rigid Propositional Designators*

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Abstract. In most modal logics, atomic propositional symbols are directly representing the meaning of sentences (such as sets of possible worlds). In other words, they use only rigid propositional designators. This means they are not able to handle uncertainty in meaning directly at the sentential level. In this paper, we offer a modal language involving non-rigid propositional designators which can also carefully distinguish *de re* and *de dicto* use of these designators. Then, we axiomatize the logics in this language with respect to all Kripke models with multiple modalities and with respect to S5 Kripke models with a single modality.

Keywords: Modal Logic · Epistemic Logic · Non-rigid Designator · Ambiguity · Propositional Quantifier

1 Introduction

We frequently fail to grasp the meaning of sentences. People who learned English only from textbooks may not get a certain contextual meaning of “this is sick!”, and anyone who is not well versed in set theory is unlikely to fully grasp even the literal meaning of “the Ultimate-L conjecture”. We also intentionally hide the meaning of symbols by designing secret interpretations of symbols to communicate private information in public: cryptographic protocols are essentially doing this, and the same string of zeros and ones can mean different things when decoded by different keys.

The famous Frege’s puzzle can also be understood in this way. To people unfamiliar with the fact that “Lewis Carroll” is the pen name of Charles Dodgson who is also a logician and responsible for Dodgson’s method in voting theory, “Lewis Carroll authored *Alice in Wonderland*” and “Charles Dodgson authored *Alice in Wonderland*” express different propositions. Indeed, they are likely to believe that the first sentence is true while the second sentence is false. However, given that Lewis Carroll is actually Charles Dodgson, “Lewis Carroll authored *Alice in Wonderland*” and “Charles Dodgson authored *Alice in Wonderland*” in fact express the same proposition.

* This work is supported by NSSF 22CZX066. The author also thanks the anonymous referees and the audience of the 2023 Beijing International Summer Workshop on Formal Philosophy for their helpful comments and suggestions.

In epistemic logic in its basic form, this ubiquitous phenomenon of uncertainty in meaning is not modeled at all. A propositional symbol p is meant to directly designate a proposition (a set of possible worlds) much like in first-order (modal) logic an individual variable x is meant to directly designate an object in the domain, and one can never be uncertain about what p means but only what's p's truth value since p is already 'interpreted'.

To our best knowledge, attempts to model uncertainty in meaning in the modal logic and possible world semantics paradigm are scarce. One notable work is [19] where different agents may interpret the same propositional symbol differently. In the usual setting of possible world semantics with multiple agents in Agt, this can be understood as taking a model to be $(W, \{R_i\}_{i\in\mathsf{Agt}}, \{V_i\}_{i\in\mathsf{Agt}})$ where R_i is the accessibility relation for i (we write the corresponding modal operator as 'B_i', 'B' for 'Belief') and for each $i \in \mathsf{Agt}$, V_i is a valuation function assigning to each propositional symbol p a set in $\wp(W)$. Then, $B_i p$ is true at a world w iff $V_i(p) \subseteq R_i(w)$; that is $B_i p$ says that i believes the proposition she takes p to mean. More generally, $B_i B_j B_k p$ means that i believes that j believes that k believes that p as interpreted by k. In other words, an occurrence of p is always interpreted by the last agent i whose belief operator scopes over that occurrence. This restriction is lifted in [18], where we can form propositional symbols p_i indexed by agent i so that p_i is always interpreted by V_i. This in a sense means that if the only uncertainty to the meaning of a propositional symbol p is how different agents may interpret it differently but unambiguously, the standard epistemic logic can simulate this by using more propositional symbols.

Another important relevant work is [17]. There, the meaning of a propositional symbol p is not merely determined by the set of possible worlds assigned to it by the valuation function V, but fundamentally by a syntactic definition $\mathsf{DEF}_w(p)$ of it using other propositional symbols, and the definition could vary from worlds to worlds. Of course, the definitions and the valuation must cohere. Then, while an agent still knows what is the proposition assigned to p by the valuation function V, the agent may not know the *definition* of p, and further the proposition expressed by the definition of p.

In this paper, we take perhaps the most straightforward way to allow uncertainty in meaning: we simply let propositional symbols be *non-rigid* designators of sets of possible worlds. In other words, we let the valuation function be world relative. This approach has been taken up in [21] to formalize definite descriptions of propositions and the Brandenburger-Keisler paradox. The paradox involves sentences such as:

(A1) Ann believes that the strangest proposition that Bob believes is that neutrinos travel at twice the speed of light.

(A2) Ann believes that the strangest proposition that Bob believes is true.

In [21], the first sentence is formalized as $B_a(\gamma \,\mathsf{is}\, \varphi)$, and the second sentence's *de dicto* and *de re* readings are formalized as $B_a^{re}\mathsf{T}(\gamma)$ and $B_a^{dicto}\mathsf{T}(\gamma)$, respectively, where γ is a definite description (non-rigid designator) for the strangest proposition that Bob believes. We find the formalism slightly cumbersome and not fully general. Taking inspiration from concept abstraction used in first-order intentional modal logic [11,13] and assignment operators used in [25,17,30,6], we relabel the syntactic category of propositional variables x which are rigid designators and use $[p/x]\varphi$ to mean "letting x be the proposition expressed by p, φ". Since the propositional variables x are only playing the role p used to play, we are only extending the basic language of modal logic by the binders $[p/x]$. With this minimal perturbation, we can already easily distinguish

- $[p/x]B_iB_jx$: letting x be the proposition p actually means, i believes that j believes that x is true;
- $B_i[p/x]B_jx$: i believes that, with x being p's meaning, j believes that x;
- $B_iB_j[p/x]x$: i believes that j believes that p is true.

The Ann and Bob sentence above can also be formalized with the help of a necessity modality $\Box$ that quantifies over all possible worlds, in which case when $\Box(x \leftrightarrow y)$ is true, x and y denote the same proposition.

- $B_a[p/x]\,\Box\,(x \leftrightarrow y)$ formalizes (A1) where y directly denotes the proposition expressed by 'neutrinos travel at twice the speed of light'.
- $[p/x]B_ax$ formalizes the *de re* reading of (A2).
- $B_a[p/x]x$ formalizes the *de dicto* reading of (A2).

The semantic type of functions from worlds to sets of worlds appears in various kinds of higher-order modal logics [15,27,12]. Indeed, there is a way to embed our language in the higher-order intentional language presented in [15]. Objects of the said type also bear the name 'two-dimensional content' and are used in for example [28,3,4,23,24]. The semantic function of the operator $[p/x]$ can also be understood as 'rigidifying' the non-rigid designator p. From this perspective, our work is related to generalized versions of hybrid logic [2]. Further discussion of relations to higher-order and hybrid modal logics are included in Section 2 after we formally introduce our minimalist language and its semantics.

Our main technical contributions are two axiomatization results, one with respect to all multiagent models, and one with respect to single-agent models where the accessibility relation is the universal relation (single-agent epistemic models). Axiomatization in our setting poses an interesting challenge that echos with the following 'paradox' on a Cantorian level: one cannot be completely ignorant of the meaning of p in the possible world framework, because there are always more possible meanings of p (sets of possible worlds) than there are possible worlds, but for different meanings X of p, we need different possible worlds to model the possibility that the agent takes X to be the meaning of p. In completeness proofs with assignment operators, one typically extends language so that in each maximally consistent set (MCS), each non-rigid designator has a witness. Let p be such a non-rigid designator. Now different MCSs should have different witnesses for p, as otherwise they are forced to take p to mean the same thing. Put in another way, we in principle need fresh witnesses for each MCS to maintain consistency when adding those witnesses. But then, we are back in Cantor's trap: no matter how we extend our language, there will always be more MCSs than there are variables. We will bypass this difficulty using step-by-step constructions.

The rest of the paper is organized as follows: in Section 2, we formally introduce the language and the semantics. We will also comment on the undecidability of the set of validities for the class of universal models (single-agent S5 case) and discuss how our language compares to higher-order and hybrid modal logics. Section 3 deals with the class of all models, i.e., the multi-agent K case, and in Section 4, we consider the class of universal models, i.e., single-agent S5 case. Finally, we conclude in Section 5 with possible future research directions.

2 Formal language and semantics

Definition 1. *We fix a countably infinite set* Prop *of propositional names, a countably infinite set* Var *of propositional variables, and a non-empty set* Agt *of unary modal operators. Then, define language* $\mathcal{L}$ *by the following grammar:*

$$\mathcal{L} \ni \varphi ::= x \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \Box\varphi \mid [p/x]\varphi$$

where $x \in \mathsf{Var}$, $p \in \mathsf{Prop}$, *and* $\Box \in \mathsf{Agt}$. *The usual abbreviations apply. Also, we treat* $[p/x]$ *as a quantifier that binds the variable* x. *Thus the usual notions of free and bound variables, free for substitution (substitutability), and so on apply as well.* $\varphi[y/x]$ *is the result of replacing all free occurrence of* x *in* φ *by* y. *We will usually accompany this notation with a substitutability requirement.*

Here symbols in Prop are non-rigid propositional designators while symbols in Var are rigid propositional designators. Syntactically we do not allow for $p \in \mathsf{Prop}$ to appear as an atomic formula since for example, $B_i B_j B_k p$ is ambiguous. Of course, we could write $B_i B_j B_k [p/x]x$ when that is the intended expression.

Definition 2. *A* Kripke model with non-rigid propositional designators *('model' for short) is a tuple* $(W, \{P_p\}_{p\in\mathsf{Prop}}, \{R_\Box\}_{\Box\in\mathsf{Agt}})$ *where*

- W *is a non-empty set, intuitively the set of possible worlds;*
- *for each* $p \in \mathsf{Prop}$, P_p *is a function from* W *to* $\wp(W)$, *with* $P_p(w)$ *understood as the proposition* p *designates at* w;
- *for each* $\Box \in \mathsf{Agt}$, $R_\Box \subseteq W^2$, *the accessibility relation for* $\Box$.

Given a model $\mathcal{M} = (W, \{P_p\}_{p\in\mathsf{Prop}}, \{R_\Box\}_{\Box\in\mathsf{Agt}})$, *an* assignment σ *for* $\mathcal{M}$ *is a function from* Var *to* $\wp(W)$. *Truth in a model* $\mathcal{M} = (W, \{P_p\}_{p\in\mathsf{Prop}}, \{R_\Box\}_{\Box\in\mathsf{Agt}})$ *is defined recursively relative to worlds and assignments as follows:*

$$\begin{aligned}
\mathcal{M}, w, \sigma \vDash x &\iff w \in \sigma(x)\\
\mathcal{M}, w, \sigma \vDash \neg\varphi &\iff \mathcal{M}, w, \sigma \nvDash \varphi\\
\mathcal{M}, w, \sigma \vDash (\varphi \wedge \psi) &\iff \mathcal{M}, w, \sigma \vDash \varphi \text{ and } \mathcal{M}, w, \sigma \vDash \psi\\
\mathcal{M}, w, \sigma \vDash \Box_i\varphi &\iff \forall v \in W, wR_\Box v \Rightarrow \mathcal{M}, v, \sigma \vDash \varphi\\
\mathcal{M}, w, \sigma \vDash [p/x]\varphi &\iff \mathcal{M}, w, \sigma[P_p(w)/x] \vDash \varphi.
\end{aligned}$$

Here $\sigma[P_p(w)/x]$ *is the function that is identical to* σ *except that* $\sigma[P_p(w)/x](x) = P_p(w)$. *This '*$f[a/x]$*' notation is used for all functions. A formula* φ *is valid on a model* $\mathcal{M}$ *if it is true at all worlds relative to all assignments (written* $\mathcal{M} \vDash \varphi$*).* φ *is valid on a class* $\mathcal{K}$ *of models if it is valid on all models in the class* $\mathcal{K}$.

The analogue of the substitution lemma in first-order logic holds as well.

Lemma 1. *For any model* $\mathcal{M} = (W, \{P_p\}_{p \in \mathsf{Prop}}, \{R_\Box\}_{\Box \in \mathsf{Agt}})$*,* $w \in W$*, assignment* σ *for* $\mathcal{M}$*, formula* $\varphi \in \mathcal{L}$*,* $x, y \in \mathsf{Var}$*, if* y *is substitutable for* x *in* φ*, then* $\mathcal{M}, w, \sigma \vDash \varphi[y/x]$ *iff* $\mathcal{M}, w, \sigma[\sigma(y)/x] \vDash \varphi$*.*

We will also be interested in the case when Agt is a singleton $\{\Box\}$, and the relation $R_\Box$ is the universal relation. Since the universal relation is uniquely determined by the set of possible worlds, we will simply dispense with it.

Definition 3. *A* universal Kripke model with non-rigid propositional designators *('universal model' for short) is a tuple* $(W, \{P_p\}_{p \in \mathsf{Prop}})$ *where* W *is a non-empty set and for each* $p \in \mathsf{Prop}$*,* $P_p : W \to \wp(W)$*. When* $\mathsf{Agt} = \{\Box\}$*, we interpreted* $\mathcal{L}$ *on universal models* $\mathcal{M} = (W, \{P_p\}_{p \in \mathsf{Prop}})$ *just like in Definition 2 except that* $\mathcal{M}, w, \sigma \vDash \Box\varphi$ *iff forall* $v \in W$*,* $\mathcal{M}, v, \sigma \vDash \varphi$*.*

These models can be used to model an S5 agent, for which the $R_\Box$ relation is an equivalence relation since truth in $\mathcal{L}$ is preserved under generated submodel. Due to lack of space, we will not define and prove this formally, but in fact, more generally, $\mathcal{L}$ can be translated into the guarded fragment, though not the two-variable fragment. Now, on universal models, the 'guard' does not really do anything, and indeed, for the class of universal models, its set of validities is undecidable. For a starter, note that:

Proposition 1. $\Box[p/x]\Diamond[p/y](\Box(x \to y) \wedge \Diamond(y \wedge \neg x))$ *is satisfiable by a universal model, and all such models are infinite.*

The idea is that $\Box[p/x]\Diamond[p/y](\Box(x \to y) \wedge \Diamond(y \wedge \neg x))$ entails there must be an infinite strictly ascending chain of sets of possible worlds. The 'paradox' mentioned in the introduction is also formalizable as $\Diamond[p/y]\Box(x \leftrightarrow y)$, and indeed no universal model can validate this formula.

Again, due to lack of space, we will not formally prove undecidability, but the idea is to use the formula $\Box[p/x](x \wedge \Box(x \to [p/y]\Box(y \leftrightarrow x)))$ so that we can use $\Box[p/x]$ to simulate the first order quantifier $\forall x$ and use another $q \in \mathsf{Prop}$ to simulate a binary relation R so that $R(x, y)$ translates to $\Diamond(x \wedge [q/z]\Diamond(y \wedge z))$. Then we can translate first-order logic with a binary relation into $\mathcal{L}$.

Now we briefly comment on how our language and semantics compare to the semantics of higher-order modal logics and hybrid logics. First, we consider the influential system **IL** (Intentional Logic) presented in [15]. As a higher-order logic, we first need to define the types of its language. To simplify the presentation, we omit the basic type e for individuals (tables and chairs) and the complex types using it. Thus, the basic type names are s and t where s names the type for possible worlds and t names the type for truth values. All types can be generated by the following BNF grammar

$$\mathsf{Type} \ni \alpha ::= t \mid (\alpha \to \alpha) \mid (s \to \alpha).$$

Note that s is not by itself a type. When parentheses are omitted, we assume right-association, e.t. $s \to s \to t$ means $(s \to (s \to t))$. For each $\alpha \in \mathsf{Type}$ we assume that there are countably infinitely many constants c and variables x of the type α (when we highlight their type, we write c_α and x_α). Then, the set T_α of the terms of type α are defined inductively by the following clauses:

- Constants and variables of type α are in T_α.
- If $A \in \mathsf{T}_{\alpha\to\beta}$ and $B \in \mathsf{T}_\alpha$, then $(AB) \in \mathsf{T}_\beta$.
- If $A \in \mathsf{T}_\beta$ and x is a variable of type α, then $(\lambda x.A) \in \mathsf{T}_{\alpha\to\beta}$.
- If $A, B \in \mathsf{T}_\alpha$, then $(A = B) \in \mathsf{T}_t$.
- If $A \in \mathsf{T}_\alpha$, then $(\hat{}A) \in \mathsf{T}_{s\to\alpha}$.
- If $A \in \mathsf{T}_{s\to\alpha}$, then $(\check{}A) \in \mathsf{T}_\alpha$.

Again, we write A_α to highlight that A is of type α and assume left-association when parentheses are omitted. Truth-functional operators such as $\neg \in \mathsf{T}_{t\to t}$ and $\wedge \in \mathsf{T}_{t\to t\to t}$ are not included as they can be defined by lambda terms, and the meaning of $\hat{}$ and $\check{}$ will become clear below.

Semantically, an object of type t is a truth value while an object of type s is understood as a possible world, and an object of type $\alpha \to \beta$ is a function from objects of type α to objects of type β. Each term A of type α extensionally denotes an object of type α and intentionally denotes an object of type $s \to \alpha$, namely a function from possible worlds to objects of type α. Thus, for a set-theoretical formal semantics, given a non-empty set W for possible worlds, we define the full domain D^W_α for each type α recursively by $D^W_t = \{0,1\}$, $D^W_s = W$, and $D^W_{\alpha\to\beta} = (D^W_\beta)^{D^W_\alpha}$, the set of all functions from D^W_α to D^W_β (here we allow α to be s). Then, a standard model for **IL** is a pair (W, I) where W is a non-empty set (of possible worlds) and I is a function that maps, for all type α, the constants c of type α to $I(c) \in D^W_{s\to\alpha}$, which we take as the intention of c in this model. An assignment σ for a model (W, I) is a function that maps each variable x of its type α to $\sigma(x) \in D^W_\alpha$. Then the denotation $|A|^{W,I,w,\sigma}$ of terms A at world w relative to assignment σ in model (W, I) is defined recursively:

- $|c|^{W,I,w,\sigma} = I(c)(w)$ and $|x|^{W,I,w,\sigma} = \sigma(x)$.
- $|AB|^{W,I,w,\sigma} = |A|^{W,I,w,\sigma}(|B|^{W,I,w,\sigma})$.
- $|\lambda x_\alpha.A_\beta|^{W,I,w,\sigma} = \{(a, |A_\beta|^{W,I,w,\sigma[a/x_\alpha]}) \mid a \in D^W_\alpha\}$.
- $|A = B|^{W,I,w,\sigma} = 1$ if $|A|^{W,I,w,\sigma} = |B|^{W,I,w,\sigma}$, and is 0 otherwise.
- $|\hat{}A|^{W,I,w,\sigma} = \{(v, |A|^{W,I,v,\sigma}) \mid v \in W\}$.
- $|\check{}A_{s\to\alpha}|^{W,I,w,\sigma} = |A_{s\to\alpha}|^{W,I,w,\sigma}(w)$.

At any world w, the idea of $\hat{}A$ is to obtain the intention of A, the total function from worlds to A's denotation at those worlds, as its denotation. Conversely, when we have a term $A_{s\to\alpha}$, its denotation is already 'intentionally of type α', and the idea of $\check{}A_{s\to\alpha}$ is to get the extension of A's denotation.

Now let us try to translate our language $\mathcal{L}$ into the language of **IL** with its standard semantics. If we were working with the basic language of propositional modal logic, then it would be a natural choice to regard $\varphi \in \mathcal{L}$ as terms of type t. This would require us to take atomic propositional symbols p as constants of type t so that they could have non-constant intentions, and take modal operators $\Box$ as terms of type $(s \to t) \to t$ as they operate on the intention of formulas. Then $\Box\varphi$

should be translated as $\Box(\hat{}\varphi)$. However, we cannot treat propositional variables x as constants of type t since syntactically we must be able to bind them. So they must be a variable of some type. The natural choice we have then is variables of type $s \to t$, since the natural translation for propositional names $p \in \mathsf{Prop}$ are constants of type $s \to t$, and $[p/x]\varphi$ can be understood as $(\lambda x.\varphi)p$. But it is also natural to take formulas as terms of type t, which coheres well with the truth-functional operators, so we must deal with the fact that in $\mathcal{L}$, propositional variables are also formulas. The solution is simple: always use $\check{}x$.

More formally, define a translation T on $\mathcal{L}$:

- $T(x) = (\check{}x_{s\to t})$ where $x_{s\to t}$ is a variable corresponding to x.
- $T(\neg\varphi) = \neg T(\varphi)$, and $T(\varphi \wedge \psi) = (\wedge T(\varphi))T(\psi)$.
- $T(\Box\varphi) = \Box_{(s\to t)\to t}(\hat{}T(\varphi))$ where $\Box_{(s\to t)\to t}$ is a constant corresponding to $\Box$.
- $T([p/x]\varphi) = (\lambda x_{s\to t}.T(\varphi))p_{s\to t}$ where $p_{s\to t}$ is a constant corresponding to p.

Then it is not hard to check that, with the obvious way to expand a model and assignment for $\mathcal{L}$ into a standard model and assignment for **IL**, φ and $T(\varphi)$ are true at precisely the same worlds.

For the hybrid way to understand $\mathcal{L}$, consider the following variation $\mathcal{L}@$ of $\mathcal{L}$ where instead of a set Var of propositional variables, we use a set Nom of nominal variables. Then $\mathcal{L}@$ is defined by the grammar

$$\varphi ::= (p@i) \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \Box\varphi \mid \downarrow i\varphi$$

where $p \in \mathsf{Prop}, i \in \mathsf{Nom}$, and $\Box \in \mathsf{Agt}$. Given a model $\mathcal{M} = (W, \{P_p\}_{p\in\mathsf{Prop}}, \{R_\Box\}_{\Box\in\mathsf{Agt}})$ and a nominal assignment $\nu : \mathsf{Nom} \to W$, we define the semantics by $\mathcal{M}, w, \nu \vDash (p@i)$ iff $w \in P_p(\nu(i))$ and $\mathcal{M}, w, \nu \vDash \downarrow i\varphi$ iff $\mathcal{M}, w, \nu[w/i] \vDash \varphi$. Then it is also not hard to see that $[p/x]\varphi$ can be understood as $\downarrow i_x\varphi[(p@i_x)/x]$ where i_x is a nominal variable corresponding to the variable x and $\varphi[(p@i_x)/x]$ is the result of replacing free occurrences of x in φ with $(p@i)$. Thus, a truth-preserving translation from sentences (formulas without free propositional variables) in $\mathcal{L}$ to sentences (formulas without free nominal variables) in $\mathcal{L}@$ can be defined.

3 Axiomatization for multi-agent K

In this section, we deal with the class of all models. Our completeness proof requires adding new variables to the language. Thus let us fix a set Var^+ that is a superset of Var and is also countably infinite. Then by $[\mathsf{Var}, \mathsf{Var}^+]$ we mean the set $\{X \mid \mathsf{Var} \subseteq X \subseteq \mathsf{Var}^+\}$.

Definition 4. *For any $X \in [\mathsf{Var}, \mathsf{Var}^+]$, define $\mathcal{L}(X)$ by the following grammar:*

$$\mathcal{L}(X) \ni \varphi ::= x \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \Box\varphi \mid [p/x]\varphi$$

where $x \in X$, $p \in \mathsf{Prop}$, and $\Box \in \mathsf{Agt}$. Obviously $\mathcal{L} = \mathcal{L}(\mathsf{Var})$.

Now we define the logic NPK (non-rigid propositional K).

Definition 5. *For any* $X \in [\mathsf{Var}, \mathsf{Var}^+]$*, let* $\mathsf{NPK}(X)$ *be the set of formulas in* $\mathcal{L}(X)$ *axiomatized by the following axioms and rules:*

- *(PL) All instances of propositional tautologies in* $\mathcal{L}(X)$
- *(K)* $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$
- *(Comm)* $[p/x](\varphi \to \psi) \to ([p/x]\varphi \to [p/x]\psi)$ *and* $[p/x]\neg\varphi \leftrightarrow \neg[p/x]\varphi$
- *(Triv)* $\varphi \leftrightarrow [p/x]\varphi$ *where* x *does not occur free in* φ
- *(Sub)* $[p/y]([p/x]\varphi \leftrightarrow \varphi[y/x])$ *whenever* y *is substitutable for* x *in* φ
- *(Perm)* $[p/x][q/y]\varphi \leftrightarrow [q/y][p/x]\varphi$ *where* x *and* y *are distinct variables*
- *(MP) from* φ *and* $\varphi \to \psi$ *derive* ψ
- *(Nec) from* φ *derive* $\Box\varphi$ *for every* $\Box \in \mathsf{Agt}$
- *(Inst) from* φ *derive* $[p/x]\varphi$

As usual, we write $\Gamma \vdash_{\mathsf{NPK}(X)} \varphi$ *to mean that* $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}(X)$ *and there is a finite conjunction* γ *of formulas in* Γ *such that* $\gamma \to \varphi$ *is in* $\mathsf{NPK}(X)$*. By* NPK *we mean* $\mathsf{NPK}(\mathsf{Var})$*.*

The soundness of these axioms and rules is easy to check, where (Sub) is the syntactic version of the substitution lemma. We collect some basic facts about the logic in the following lemma:

Lemma 2. *Let* $X, Y \in [\mathsf{Var}, \mathsf{Var}^+]$ *such that* $X \subseteq Y$*.*

- $\mathsf{NPK}(X)$ *proves equivalence under renaming of bound variables.*
- *If* $\Gamma \subseteq \mathcal{L}(X)$ *is consistent in* $\mathsf{NPK}(X)$ *(that is,* $\Gamma \nvdash_{\mathsf{NPK}(X)} \bot$*), then there is a maximally consistent set (MCS for short)* Δ *w.r.t.* $\mathsf{NPK}(X)$ *extending* Γ*. We use choice to fix such a set uniformly as* $Ext_{\mathsf{NPK}(X)}(\Gamma)$*.*
- *For any* $\Gamma \subseteq \mathcal{L}(X)$ *and any* $\Box \in \mathsf{Agt}$*, define* $\Box^{-1}\Gamma = \{\varphi \mid \Box\varphi \in \Gamma\}$*. Then if* Γ *is consistent in* $\mathsf{NPK}(X)$*, for any formula* $\neg\Box\varphi \in \Gamma$*,* $\{\neg\varphi\} \cup \Box^{-1}\Gamma$ *is consistent.*
- $\mathsf{NPK}(Y)$ *is conservative over* $\mathsf{NPK}(X)$*:* $\mathsf{NPK}(Y) \cap \mathcal{L}(X) = \mathsf{NPK}(X)$*.*

The first three points are standard exercises. For the last point, note that any proof in $\mathsf{NPK}(Y)$ uses only finitely many variables. Thus we can always find unused variables in X and uniformly replace variables in $Y \setminus X$ used in the proof by these new variables in X.

Definition 6. *A* witness assignment *is an injective function* v *from* Prop *to* Var^+*. We often write* v_p *for* $v(p)$*. For any* $X \in [\mathsf{Var}, \mathsf{Var}^+]$*, a witness assignment* v *is* fresh for X *if* $ran(v) \cap X = \varnothing$*. For any witness assignment* v *and* $X \in [\mathsf{Var}, \mathsf{Var}^+]$*, define* $\mathrm{WF}(v, X)$ *(witnessing formulas in* $\mathcal{L}(X)$ *using* v*) to be*

$$\{[p/x]\alpha \leftrightarrow \alpha[v_p/x] \mid [p/x]\alpha \in \mathcal{L}(X), v_p \text{ is substitutable for } x \text{ in } \alpha\}.$$

We also write $X + v$ *for* $X \cup ran(v)$*.*

Lemma 3. *For any* $X \in [\mathsf{Var}, \mathsf{Var}^+]$*,* Γ *a MCS in* $\mathsf{NPK}(X)$*, and* v *a witness assignment fresh for* X*, the set* $\Gamma' = \Gamma \cup \mathrm{WF}(v, X+v)$ *is consistent in* $\mathsf{NPK}(X+v)$ *and has exactly one MCS extension in* $\mathsf{NPK}(X+v)$*. We denote this extension by* $\Gamma + v$*.*

Proof. In this proof, we write $\vdash$ for $\vdash_{\mathsf{NPK}(X)}$ and $\vdash_{+v}$ for $\vdash_{\mathsf{NPK}(X+v)}$. First, we show consistency. Suppose not, then we have a finite $\{\alpha_1, \dots, \alpha_n\} \subseteq \Gamma$ and a finite $\{[p_i/x_i]\beta_i \leftrightarrow \beta_i[v_{p_i}/x_i]\}_{i=1}^m \subseteq \mathrm{WF}(v, X+v)$ with

$$\vdash_{+v} \bigwedge_i \alpha_i \to \neg \bigwedge_i ([p_i/x_i]\beta_i \leftrightarrow \beta_i[v_{p_i}/x_i]). \tag{1}$$

Now we have the following derivable formulas:

$$\vdash_{+v} \bigwedge \alpha_i \to \neg \bigwedge_i [p_1/v_{p_1}] \dots [p_m/v_{p_m}]([p_i/x_i]\beta_i \leftrightarrow \beta_i[v_{p_i}/x_i]). \tag{2}$$

$$\vdash_{+v} [p_i/v_{p_i}]([p_i/x_i]\beta_i \leftrightarrow \beta_i[v_{p_i}/x_i]). \tag{3}$$

$$\vdash_{+v} [p_1/v_{p_1}] \dots [p_m/v_{p_m}]([p_i/x_i]\beta_i \leftrightarrow \beta_i[v_{p_i}/x_i]). \tag{4}$$

$$\vdash_{+v} \bigwedge \alpha_i \to \bigwedge_i [p_1/v_{p_1}] \dots [p_m/v_{p_m}]([p_i/x_i]\beta_i \leftrightarrow \beta_i[v_{p_i}/x_i]). \tag{5}$$

(2) is obtained from (1) by repeated use of (Inst), (Comm), and (Triv). (3) are simply instances of (Sub). (4) are obtained from (3) by (Inst) and (Perm). (5) is simply combining (4) for all i and add an antecedent. Thus, Γ is inconsistent in $\mathsf{NPK}(X+v)$. By the conservativity of $\mathsf{NPK}(X+v)$ over $\mathsf{NPK}(X)$, Γ is inconsistent in $\mathsf{NPK}(X)$, contradicting the assumption.

Now we show that Γ' has at most one maximally consistent extension in $\mathsf{NPK}(X+v)$. For this, it is enough to show that for any $\varphi \in \mathcal{L}(X+v)$, if $\Gamma' \nvdash_{+v} \varphi$, then $\Gamma' \vdash_{+v} \neg\varphi$. So suppose $\Gamma' \nvdash_{+v} \varphi$. Let ψ be the result of renaming bound variables in φ so that all bound variables are in X. Then $\Gamma' \nvdash_{+v} \psi$ as $\mathsf{NPK}(X+v)$ proves equivalence under such renamings (and renamings are reversible by renamings again). Now list the free variables of ψ in $ran(v)$ as $v_{p_1}, v_{p_2}, \dots, v_{p_l}$ and pick distinct variables $x_1, x_2, \dots, x_l$ in X that does not appear in ψ. Then inductively define the formulas $\alpha_0 = \psi$, $\alpha_{i+1} = [p_{i+1}/x_{i+1}](\alpha_i[x_{i+1}/v_{p_{i+1}}])$. Then α_l is in fact $[p_l/x_l] \dots [p_1/x_1](\psi[x_1/v_{p_1}] \dots [x_l/v_{p_l}])$ and moreover, for each $i = 0 \dots l-1$, $\alpha_{i+1} \leftrightarrow \alpha_i$ is in Γ', since $\alpha_i[x_{i+1}/v_{p_{i+1}}][v_{p_{i+1}}/x_{i+1}]$ is identical to α_i and hence $\alpha_{i+1} \leftrightarrow \alpha_i$ is in the form of $[p/x]\beta \leftrightarrow \beta[v_p/x]$. Thus $\Gamma' \nvdash_{+v} \alpha_l$. But now $\alpha_l \in \mathcal{L}(X)$, so by the maximality of Γ, $\Gamma' \vdash_{v+} \neg\alpha_l$. By (Comm),

$$\Gamma' \vdash_{v+} [p_l/x_l] \dots [p_1/x_1](\neg\psi[x_1/v_{p_1}] \dots [x_l/v_{p_l}])$$

Then by using formulas in $\mathrm{WF}(v, X)$, we see that $\Gamma' \vdash_{+v} \neg\psi$.

Let Γ be a maximally consistent set for NPK. To prepare for the model building for Γ, first pick for each $i \in \mathbb{N}$ a witness assignment v^i such that for any $i \neq j$, $ran(v^i) \cap ran(v^j) = \varnothing$ and $ran(v^i) \cap \mathsf{Var} = \varnothing$. Also, let $\mathsf{Var}_0 = \mathsf{Var} + v^0$ and $\mathsf{Var}_{i+1} = \mathsf{Var}_i + v^{i+1}$. Then let $\mathsf{Var}_\omega = \bigcup_{i \in \mathbb{N}} \mathsf{Var}_i$.

Now we construct a tree model for Γ in stages. Each node of the tree is of the form (s, Δ) where s is a sequence of modal operators in Agt and Δ is a MCS in $\mathcal{L}(\mathsf{Var}_{len(s)})$ such that $WF(v^{len(s)}, \mathsf{Var}_{len(s)}) \subseteq \Delta$. $len(s)$ is the length of s.

At stage 0, the tree is $T_0 = \{(\epsilon, \Gamma + v^0))\}$ where ϵ is the empty sequence. Then inductively, we define T_{i+1} as the result of adding to T_i for each leaf node (s, Δ) (it is a leaf in the sense that $len(s) = i$), for each $\Box \in \mathsf{Agt}$, and for each formula in Δ of the form $\neg\Box\varphi$, the pair $(s+\Box, Ext_{\mathsf{NPK}(\mathsf{Var}_i)}(\{\neg\varphi\}\cup\Box^{-1}(\Delta))+v^{i+1})$. Here $s+\Box$ is the sequence that extends s by $\Box$. Finally, set $\mathcal{M}^{\mathsf{NPK}}=(T,\{P_p\}_{p\in\mathsf{Prop}},\{R_\Box\}_{\Box\in\mathsf{Agt}})$ where:

- $T=\bigcup_{i\in\mathbb{N}}T_i$;
- $(s_1,\Delta_1)R_\Box(s_2,\Delta_2)$ iff $s_2=s_1+\Box$
- $P_p((s,\Delta))=\{(s',\Delta')\in T\mid v_p^{len(s)}\in\Delta'\}$.

Definition 7. *A* formal assignment *g for $\mathcal{M}^{\mathsf{NPK}}$ is a function from Var to Var_ω such that $g(x)$ is either x itself or is in $\mathsf{Var}_\omega\setminus\mathsf{Var}$. We extend g so that for any $\varphi\in\mathcal{L}$, $g(\varphi)=\varphi[g(x)/x]$. Note that $g(x)$ is always substitutable for x in φ. For each formal assignment g for $\mathcal{M}^{\mathsf{NPK}}$, define assignment $\overline{g}$ by*

$$\overline{g}(x)=\{(s,\Delta)\in T\mid g(x)\in\Delta\}.$$

Also, for each $(s,\Delta)\in T$, we say that a formal assignment g is admissible *for (s,Δ) if $ran(g)\subseteq\mathsf{Var}_{len(s)}$.*

Lemma 4. *For any formula $\varphi\in\mathcal{L}$, any formal assignment g for $\mathcal{M}^{\mathsf{NPK}}$, and any $(s,\Delta)\in T$, if g is admissible for (s,Δ), then $\mathcal{M}^{\mathsf{NPK}},(s,\Delta),\overline{g}\vDash\varphi$ iff $g(\varphi)\in\Delta$.*

Proof. For the base case, note that for any $x\in\mathsf{Var}$, trivially by definition,

$$\mathcal{M}^{\mathsf{NPK}},(s,\Delta),\overline{g}\vDash x\Leftrightarrow(s,\Delta)\in\overline{g}(x)\Leftrightarrow g(x)\in\Delta.$$

For the Boolean cases, we only need to note that $g(\neg\alpha)=\neg g(\alpha)$ and $g((\alpha\wedge\beta))=(g(\alpha)\wedge g(\beta))$ and that Δ is maximally consistent.

For one direction of the modal cases, suppose $g(\Box\varphi)\in\Delta$. Then $\Box g(\varphi)\in\Delta$. By the construction of T, for any $(s+\Box,\Delta')\in T$, $g(\varphi)\in\Delta'$. By Induction Hypothesis (IH), and noting that since g is admissible for (s,Δ), g must also be admissible for $(s+\Box,\Delta')$, $\mathcal{M}^{\mathsf{NPK}},(s+\Box,\Delta'),\overline{g}\vDash\varphi$. By the definition of $R_\Box$, $\mathcal{M}^{\mathsf{NPK}},(s,\Delta),\overline{g}\vDash\Box\varphi$.

For the other direction of the modal cases, suppose $g(\Box\varphi)\notin\Delta$. Since g is admissible for (s,Δ), $g(\Box\varphi)\in\mathcal{L}(\mathsf{Var}_{len(s)})$. Since Δ is a MCS of $\mathsf{NPK}(\mathsf{Var}_{len(s)})$, $\neg\Box g(\varphi)\in\Delta$. By the construction of T, there is $(s+\Box,\Delta')\in T$ such that $\neg g(\varphi)\in\Delta'$, and then $g(\varphi)\notin\Delta'$. By IH, $\mathcal{M}^{\mathsf{NPK}},(s+\Box,\Delta'),\overline{g}\nvDash\varphi$, and thus $\mathcal{M}^{\mathsf{NPK}},(s,\Delta),\overline{g}\nvDash\Box\varphi$.

Finally, for the assignment operator case, consider any formula $[p/x]\varphi\in\mathcal{L}$.

$$\mathcal{M}^{\mathsf{NPK}},(s,\Delta),\overline{g}\vDash[p/x]\varphi\Leftrightarrow\mathcal{M}^{\mathsf{NPK}},(s,\Delta),\overline{g[v_p^{len(s)}/x]}\vDash\varphi\Leftrightarrow g[v_p^{len(s)}/x](\varphi)\in\Delta.$$

The first equivalence is due to our definition of P_p, $\overline{g}[P_p(s,\Delta)/x]=\overline{g[v_p^{len(s)}/x]}$, and the second is by IH. Observe that the formula $g([p/x]\varphi)\leftrightarrow g[v_p^{len(s)}/x](\varphi)$ is precisely of the form $[p/x]\beta\leftrightarrow\beta[v_p^{len(s)}/x]\in\mathrm{WF}(v^{len(s)},\mathsf{Var}_{len(s)})$, where β is the result of replacing each free variable $y\neq x$ in φ by $g(y)$. Since g is admissible for Δ so that $\beta\in\mathcal{L}(\mathsf{Var}_{len(s)})$ and by construction Δ is of the form $\Xi+v^{len(s)}$ where Ξ is maximally consistent, the formula $g([p/x]\varphi)\leftrightarrow g[v_p^{len(s)}/x](\varphi)$ is in Δ. Thus $g[v_p^{len(s)}/x]\varphi\in\Delta$ iff $g([p/x]\varphi)\in\Delta$.

Given the last truth lemma, $\mathcal{M}^{\mathsf{NPK}}, (\epsilon, \Gamma + v^0), \overline{id}$ satisfies Γ, where id is the identity function from Var to Var. Thus,

Theorem 1. NPK *is sound and strongly complete with respect to the class of all Kripke models with non-rigid propositional designators.*

4 Axiomatization for single agent S5

In this section, we deal with the case where $\mathsf{Agt} = \{\Box\}$ and models are universal. To facilitate describing a special axiom for S5, where $\vec{p} = (p_1, \ldots, p_n)$ is a finite sequence from Prop of length $n \in \mathbb{N}^+$, and $\vec{x} = (x_1, \ldots, x_n)$ is a finite sequence from Var^+ of equal length n, by $[\vec{p}/\vec{x}]$ we mean the stack of assignment operators $[p_1/x_1]\cdots[p_n/x_n]$. Also, when v is an injective function from Prop to Var^+, by $v_{\vec{p}}$ we mean the sequence $(v_{p_1}, \ldots, v_{p_n})$. Thus $[\vec{p}/v_{\vec{p}}]$ is $[p_1/v_{p_1}]\cdots[p_n/v_{p_n}]$.

Definition 8. *For any* $X \in [\mathsf{Var}, \mathsf{Var}^+]$*, let* $\mathsf{NPS5}(X)$ *be the set of formulas in* $\mathcal{L}(X)$ *axiomatized by all the axioms and rules defining* $\mathsf{NPK}(X)$ *and also:*

- *All instances of the usual S5 axioms.*
- *(SymSub)* $[\vec{p}/v_{\vec{p}}](\gamma \to \Box[\vec{q}/\vec{z}] \Diamond (\gamma \land \bigwedge_{i=1}^{m}([p_i/x_i]\varphi_i \leftrightarrow \varphi_i[v_{p_i}/x_i])))$ *where* $\vec{p} = (p_1, \ldots, p_n)$ *is from* Prop*,* v *is an injection from* Prop *to* X*,* $\vec{q}$ *and* $\vec{z}$ *are sequences of equal length from* Prop *and* X *respectively, variables in* $\vec{z}$ *does not occur in* γ *or* $v_{\vec{p}}$*, and* v_{p_i} *is substitutable for* x_i *in* φ_i*.*

(SymSub) says something stronger than (Sub): under the assignment $[p/y]$, even if some other variable z in φ is bound by the value of p at some other world, still $[p/x]\varphi \leftrightarrow \varphi[y/x]$ at this world. We can return to 'this world' by $\Box\Diamond$ since the underlying accessibility relation is universal (and hence symmetric). The extra formula γ further solidifies that we are returning to 'this world'. Then, the soundness of these axioms and rules over universal models is not hard to check.

The analogue of Lemma 2 and Lemma 3 holds also for $\mathsf{NPS5}$ since they only use the NPK part, and for the lack of space we do not repeat then here. The following technical lemma shows the use of (SymSub).

Lemma 5. *Suppose* $X \in [\mathsf{Var}, \mathsf{Var}^+]$*,* v^1 *is a witness assignment such that* $ran(v^1) \subseteq X$*,* v^2 *is a witness assignment fresh for* X*,* Γ_1 *and* Γ_2 *are both MCSs in* $\mathsf{NPS5}(X)$ *such that* $\mathrm{WF}(v^1, X) \subseteq \Gamma_1$ *and* $\Box^{-1}(\Gamma_1) \subseteq \Gamma_2$*, and finally* $\Delta_2 = \Gamma_2 + v^2$*. Then* $\Gamma_1 \cup \mathrm{WF}(v^1, X + v^2) \cup \Box^{-1}\Delta_2$ *is consistent in* $\mathsf{NPS5}(X + v^2)$*.*

Proof. We write $\vdash$ for $\vdash_{\mathsf{NPS5}(X)}$ and $\vdash_{+v^2}$ for $\vdash_{\mathsf{NPS5}(X+v^2)}$. Suppose toward a contradiction that $\Gamma_1 \cup \mathrm{WF}(v^1, X + v^2) \cup \Box^{-1}\Delta_2$ is inconsistent. Since Γ_1 and $\Box^{-1}\Delta_2$ are closed under conjunctions, there are $\gamma \in \Gamma_1$, $\delta \in \Box^{-1}\Delta_2$, and formulas $[p_i/x_i]\varphi_i \leftrightarrow \varphi_i[v^1_{p_i}/x_i]$ $(i = 1 \ldots m)$ from $\mathrm{WF}(v^1, X + v^2)$ such that

$$\gamma, \bigwedge_{i=1}^{m}([p_i/x_i]\varphi_i \leftrightarrow \varphi_i[v^1_{p_i}/x_i]), \delta \vdash_{+v^2} \bot.$$

Since $\mathsf{NPS5}$ proves equivalence under renaming bound variables, without loss of generality, we can assume that all the bound variables in all the φ_i appear in X. By Boolean and normal modal reasoning, we have

$$\delta \vdash_{+v^2} \gamma \to \neg \bigwedge_{i=1}^{m}([p_i/x_i]\varphi_i \leftrightarrow \varphi_i[v^1_{p_i}/x_i]),$$

$$\Box\delta \vdash_{+v^2} \Box(\gamma \to \neg \bigwedge_{i=1}^{m}([p_i/x_i]\varphi_i \leftrightarrow \varphi_i[v^1_{p_i}/x_i])),$$

$$\Box\delta \vdash_{+v^2} \neg \Diamond (\gamma \wedge \bigwedge_{i=1}^{m}([p_i/x_i]\varphi_i \leftrightarrow \varphi_i[v^1_{p_i}/x_i])).$$

Since $\delta \in \Box^{-1}\Delta_2$, this means that $\neg \Diamond (\gamma \wedge \bigwedge_{i=1}^{m}([p_i/x_i]\varphi_i \leftrightarrow \varphi_i[v^1_{p_i}/x_i]))$ is also in Δ_2. We will show that

$$\Diamond(\gamma \wedge \bigwedge_{i=1}^{m}([p_i/x_i]\varphi_i \leftrightarrow \varphi_i[v^1_{p_i}/x_i])) \qquad (\beta)$$

is also in Δ_2, rendering Δ_2 inconsistent. Since Γ_2 is consistent, by Lemma 3, Δ_2 should also be consistent, a contradiction.

Enumerate the set $\{p \in \mathsf{Prop} \mid v^2_p \text{ occurs in } \beta\}$ as $\vec{q} = (q_1, \dots, q_l)$. Then pick fresh variables $\vec{z} = (z_1, \dots, z_l)$ from X and let $\psi_i = \varphi_i[z_1/v^2_{q_1}]\cdots[z_l/v^2_{q_l}]$ for each $i = 1 \dots m$. Note that since γ is from Γ_1, no variables in $ran(v^2)$ occurs in γ. Now consider the formula

$$[\vec{p}/v^1_{\vec{p}}](\gamma \to \Box[\vec{q}/\vec{z}] \Diamond (\gamma \wedge \bigwedge_{i=1}^{m}([p_i/x_i]\psi_i \leftrightarrow \psi_i[v^1_{p_i}/x_i]))).$$

Note that this is in the form of the axiom (SymSub) and is in $\mathcal{L}(X)$. Thus it is in Γ_1. But since $\mathrm{WF}(v^1, X) \subseteq \Gamma_1$, $\gamma \to \Box[\vec{q}/\vec{z}] \Diamond (\gamma \wedge \bigwedge_{i=1}^{m}([p_i/x_i]\psi_i \leftrightarrow \psi_i[v^1_{p_i}/x_i]))$ is in Γ_1. Since $\gamma \in \Gamma_1$, $\Box[\vec{q}/\vec{z}] \Diamond (\gamma \wedge \bigwedge_{i=1}^{m}([p_i/x_i]\psi_i \leftrightarrow \psi_i[v^1_{p_i}/x_i]))$ is also in Γ_1. Since $\Box^{-1}\Gamma_1 \subseteq \Gamma_2$,

$$[\vec{q}/\vec{z}] \Diamond (\gamma \wedge \bigwedge_{i=1}^{m}([p_i/x_i]\psi_i \leftrightarrow \psi_i[v^1_{p_i}/x_i])) \qquad (\alpha)$$

is in Γ_2, and hence is in $\Delta_2 = \Gamma_2 + v^2$. But observe that β can be obtained by iteratively removing $[q_i/z_i]$ and instantiate z_i with $v^2_{q_i}$ (reversing the process of constructing ψ_i from φ_i). Since $\Delta_2 = \Gamma_2 + v^2$, $\mathrm{WF}(v^2, X + v^2) \subseteq \Delta_2$. So Δ_2 proves $\alpha \leftrightarrow \beta$, and hence β is in Δ_2.

Let Γ be a maximally consistent set for $\mathsf{NPS5}$. To build a universal model for Γ, pick fresh witness assignments $\{v^i \mid i \in \mathbb{N}\}$ and corresponding variable sets Var_i and Var_ω as before. It is useful to note that $\mathcal{L}(\mathsf{Var}_\omega) = \bigcup_{i\in\mathbb{N}} \mathcal{L}(\mathsf{Var}_i)$ since each formula is finite and uses only finitely many variables in Var_ω. We fix an enumeration of $(\neg \Box \chi_1, \neg \Box \chi_2, \dots)$ of all formulas in $\mathcal{L}(\mathsf{Var}_\omega)$ of the form $\neg \Box \varphi$.

Now we build a model for Γ in stages, where at stage i, we build a sequence $\Sigma^i = (\Sigma^i_0, \dots, \Sigma^i_i)$ of MCSs in $\mathsf{NPS5}(\mathsf{Var}_i)$, a set $\Pi^i \subseteq \mathcal{L}(\mathsf{Var}_i)$, and a formula $\neg \Box \theta_i$ (here $i > 0$) with set $H^i = \{\neg \Box \theta_1, \dots, \neg \Box \theta_i\}$ (with $H^0 = \varnothing$) such that:

- for each $j = 0 \dots i$, $WF(v^j, \mathsf{Var}_i) \subseteq \Sigma^i_j$;
- for each $j = 0 \dots i$, $\Box^{-1}\Sigma^i_j = \Pi^i$;
- for each $j = 1 \dots i$, $\neg\theta_j \in \Sigma^i_j$ and $\neg\Box\theta_j$ is the first formula in the sequence $(\neg\Box\chi_1, \neg\Box\chi_2, \dots)$ that appears in $\Pi^{j-1} \setminus H^{j-1}$.

Intuitively, Π^i is the 'modal theory' of the model at stage i, and H^i is the set of $\neg\Box$ formulas processed before and at stage i.

We start the process with $\Sigma^0 = (\Sigma^0_0)$ where $\Sigma^0_0 = \gamma + v^0$ and $\Pi^0 = \Box^{-1}\Sigma^0_0$. Then, inductively for each $i \in \mathbb{N}$, we define Σ^{i+1} and Π^{i+1} as follows:

- Let $\neg\Box\theta_{i+1}$ be the first in $(\neg\Box\chi_1, \neg\Box\chi_2, \dots)$ that is in $\Pi^i \setminus H^i$. There must be one since H^i is finite while using redundant conjuncts, there are infinitely many formulas of the form $\neg\Box\varphi$ in Π^i.
- Since $\Pi^i = \Box^{-1}\Sigma^i_i$, by (S5), $\Pi^i \cup \{\neg\theta_{i+1}\}$ is consistent in $\mathsf{NPS5}(\mathsf{Var}_i)$. Let $\Sigma^{i+1}_{i+1} = Ext_{\mathsf{NPS5}(\mathsf{Var}_i)}(\Pi^i \cup \{\neg\theta_{i+1}\}) + v^{i+1}$. Then let $\Pi^{i+1} = \Box^{-1}\Sigma^{i+1}_{i+1}$.
- For each $j = 0 \dots i$, by construction $WF(v^j, \mathsf{Var}_i) \subseteq \Sigma^i_j$. Also, since $\Box^{-1}(\Sigma^i_j) = \Pi^i$, $\Box^{-1}(\Sigma^i_j) \subseteq Ext_{\mathsf{NPS5}(\mathsf{Var}_i)}(\Pi^i \cup \{\neg\theta_{i+1}\})$. Moreover, $\mathsf{Var}_{i+1} = \mathsf{Var}_i + v^{i+1}$. Thus, Lemma 5 applies, and $\Sigma^i_j \cup WF(v^j, \mathsf{Var}_{i+1}) \cup \Pi^{i+1}$ is consistent. We let $\Sigma^{i+1}_j = Ext_{\mathsf{NPS5}(\mathsf{Var}_{i+1})}(\Sigma^i_j \cup WF(v^j, \mathsf{Var}_{i+1}) \cup \Pi^{i+1})$. By (S5), $\Box^{-1}\Sigma^{i+1}_j = \Pi^{i+1}$.

Now we combine the sequences into a single model. For each $i \in \mathbb{N}$, let $\Delta_i = \bigcup_{j \geq i} \Sigma^j_i$. We also set $\Pi = \bigcup_{i \in \mathbb{N}} \Pi^i$ and $H = \bigcup_{i \geq 1} H^i$, which is $\{\neg\Box\theta_1, \neg\Box\theta_2, \dots\}$.

Lemma 6. *For each $i \in \mathbb{N}$: Δ_i is a MCS of $\mathsf{NPS5}(\mathsf{Var}_\omega)$, $WF(v^i, \mathsf{Var}_\omega) \subseteq \Delta_i$, and $\Box^{-1}\Delta_i = \Pi$. Moreover, for any formula of the form $\neg\Box\varphi \in \Pi$, there is $i \in \mathbb{N}$ such that $\neg\varphi \in \Delta_i$.*

Proof. Since each Σ^j_i is a MCS of $\mathsf{NPS5}(\mathsf{Var}_j)$ and $(\Sigma^j_i)_{j \geq i}$ is also an ascending sequence, Δ_i is a MCS of $\mathsf{NPS5}(\mathsf{Var}_\omega)$. Also, for each $j \geq i$, $WF(v^i, \mathsf{Var}_j) \subseteq \Sigma^j_i \subseteq \Delta_i$. This means $WF(v^i, \mathsf{Var}_\omega) \subseteq \Delta_i$. The proof for $\Box^{-1}\Delta_i = \Pi$ is also not hard.

Now take any formula $\neg\Box\varphi \in \Pi$. Let i be the smallest such that $\neg\Box\varphi \in \Pi^i$, and also let j be such that $\neg\Box\varphi = \neg\Box\chi_j$. Then by construction, $\neg\Box\varphi$ must be in H^{i+j} since at every stage after i, a formula $\neg\Box\chi_k$ before $\neg\Box\chi_j$ must be processed if $\neg\Box\chi_j$ is not processed at that stage. This means there $k \leq i + j$ such that $\neg\varphi \in \Sigma^k_k \subseteq \Delta_k$.

Given the above lemma, we define $\mathcal{M}^{\mathsf{NPS5}} = (D, \{P_p\}_{p \in \mathsf{Prop}})$ where $D = \{\Delta_i \mid i \in \mathbb{N}\}$ and for any $\Delta_i \in D$, $P_p(\Delta_i) = \{\Delta_j \in D \mid v^i_p \in \Delta_j\}$. Then, similar to Definition 7, a formal assignment g for $\mathcal{M}^{\mathsf{NPS5}}$ is a function from Var to Var_ω such that $g(x)$ is either x itself or is not in Var. Then for any $\varphi \in \mathcal{L}$, $g(\varphi) = \varphi[g(x)/x]$. Further, define the corresponding assignment $\overline{g}$ for $\mathcal{M}^{\mathsf{NPS5}}$ by $\overline{g}(x) = \{\Delta_j \in D \mid g(x) \in \Delta_j\}$. The concept of admissibility is not needed here.

Lemma 7. *For any formula $\varphi \in \mathcal{L}$, any formal assignment g for $\mathcal{M}^{\mathsf{NPS5}}$, and any $\Delta_i \in D$, $\mathcal{M}^{\mathsf{NPS5}}, \Delta_i, \overline{g} \vDash \varphi$ iff $g(\varphi) \in \Delta_i$.*

Proof. The base case and the Boolean cases are again easy. For the modal case, if $g(\Box\varphi) = \Box g(\varphi) \in \Delta_i$, then by S5 logic, $\Box\Box g(\varphi) \in \Delta_i$. Then $\Box g(\varphi) \in \Pi$ and

$g(\varphi) \in \Delta_j$ for any $\Delta_j \in D$ by Lemma 6. By IH, for any $\Delta_j \in D$, $\mathcal{M}^{\mathsf{NPS5}}, \Delta_j, \overline{g} \vDash \varphi$. Then $\mathcal{M}^{\mathsf{NPS5}}, \Delta_i, \overline{g} \vDash \Box\varphi$.

If $g(\Box\varphi) = \Box g(\varphi) \notin \Delta_i$, then by maximality, $\neg \Box g(\varphi) \in \Delta_i$. By S5 logic, $\Box\neg \Box g(\varphi) \in \Delta_i$, and $\neg \Box g(\varphi) \in \Pi$. By Lemma 6, there is $\Delta_j \in D$ such that $\neg g(\varphi) \in \Delta_j$. By consistency and IH, $\mathcal{M}^{\mathsf{NPS5}}, \Delta_j, \overline{g} \nvDash \varphi$. Then $\mathcal{M}^{\mathsf{NPS5}}, \Delta_i, \overline{g} \nvDash \Box\varphi$.

Finally, for the assignment operator case, consider any formula $[p/x]\varphi \in \mathcal{L}$. Now because $\overline{g}[P_p(\Delta_i)/x] = \overline{g[v_p^i/x]}$ and IH,

$$\mathcal{M}^{\mathsf{NPS5}}, \Delta_i, \overline{g} \vDash [p/x]\varphi \Leftrightarrow \mathcal{M}^{\mathsf{NPS5}}, \Delta_i, \overline{g[v_p^i/x]} \vDash \varphi \Leftrightarrow g[v_p^i/x](\varphi) \in \Delta_i.$$

Then, noting that $g([p/x]\varphi) \leftrightarrow g[v_p^i/x](\varphi)$ is in $\mathrm{WF}(v^i, \mathsf{Var}_\omega) \subseteq \Delta_i$, $\mathcal{M}^{\mathsf{NPS5}}, \Delta_i, \overline{g} \vDash [p/x]\varphi \Leftrightarrow g([p/x]\varphi) \in \Delta_i$.

By the above truth lemma, we have

Theorem 2. NPS5 *is sound and strongly complete with respect to the class of all universal Kripke models with non-rigid propositional designators.*

5 Conclusion

We have only scratched the surface of the formalism proposed in this paper. The immediate next step is to consider the logic of multi-agent epistemic models, be it with equivalence relations, transitive relations, or some other special relations of interest, since only then can we start talking about uncertainty in meaning in a multi-agent setting, and consider the information dynamics on the meaning of sentences. Models with a universal modality are also very important since the universal modality can help us express equality between propositions. We believe that by combining our two constructions in this paper, axiomatizations can be obtained in most of the cases.

Once we start working in a multi-agent epistemic setting, the ideas in [18,19] and [17] are worth incorporating. When talking about how different people may interpret p differently, an obvious drawback of our semantics is that there is always a ground truth of what p actually means. But in many situations, the meaning of p may be completely relative (before a convention is reached). In that case, the best we can do is to have versions p_i of p for each agent i, representing how agent i interprets p, just like in [18]. But importantly, and differently from [18], each p_i is still non-rigid, since an agent i may well be uncertain how j interprets p, i.e., what p_j means. The issue of definition in [17] can also be discussed in our framework. For example, when $\Box$ is the universal modality, $\Box[p/x][q/y][r/z]\Box(x \leftrightarrow (y \wedge z))$ seems to say that p is defined by $q \wedge r$. Of course, one may take this as only saying that p and $q \wedge r$ are necessarily equal in the proposition expressed, and definitions are more hyperintentional than that. But our framework is already hyperintentional in a sense: if we take functions from worlds to truth values, namely sets of possible worlds, as intentional, then functions from worlds to sets of possible worlds seem deserving of the description 'hyperintentional'. A discussion of how our framework relates to other hyperintentional frameworks such as [26,29,23,24] is needed here. Another important addition to consider is information dynamics. Since p is now non-rigid, updating with p relates to externalism in epistemology [9,5,16].

The extra axiom (SymSub) may look unseemly to many. We see two possible ways to eliminate it. The first is through nominals [1]: then we believe the axiom can be replaced by $[p/y](i \to \Box[q/z]\Box(i \to ([p/x]\varphi \leftrightarrow \varphi[y/x])))$. If we use $(p@i)$ and $\downarrow i$ as in $\mathcal{L}@$ mentioned in Section 2, then as the assignment operator can be eliminated, a simple axiomatic system may be obtained. Another way is by introducing propositional quantifiers $\forall x$ binding propositional variables $x \in \mathsf{Var}$ [10] since what we really want is $[p/y]\forall z(\varphi \leftrightarrow \varphi[y/x])$ where z could range over propositions denoted by some q at other worlds. A Barcan formula $\forall x \Box\varphi \to \Box\forall x\varphi$ and an instantiation axiom $\forall x\varphi \to [p/x]\varphi$ intuitively correspond to the minimal requirement on the range of propositional variables. Note that if we insist that the semantics of $\forall x$ considers all sets of possible worlds, we will immediately run into non-axiomatizability even in single-agent cases, unlike in situations without non-rigid propositional designators and assignment operators [20,7,8], since those non-rigid designators can simulate arbitrary modal operators, and results such as [22,14] would apply. But without this 'full domain' requirement, we believe axiomatizations are within reach. Generalizing to an algebraic setting that can avoid assuming that there are possible worlds (world propositions) may also be interesting. Here it may be useful to interpret $[p/x]\varphi$ as $\forall x([p]x \to \varphi)$ where for each $p \in \mathsf{Prop}$, $[p]$ is a unary modality so that $[p]x$ means 'what p means is x'. This essentially goes back to the expression $(\gamma \,\mathsf{is}\, \varphi)$ used in [21]. An assumption we have made throughout the paper is that there is always a unique proposition meant by p. This can be expressed by $\exists x([p]x \wedge \forall y([p]y \to (x = y)))$ using equality between propositions. It remains to be seen what axiomatizability results follow from this setting.

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