Some General Completeness Results for Propositionally Quantified Modal Logics

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Abstract

We study the completeness problem for propositionally quantified modal logics on quantifiable general frames, where the admissible sets are the propositions the quantifiers can range over and expressible sets of worlds are admissible, and Kripke frames, where the quantifiers range over all sets of worlds. We show that any normal propositionally quantified modal logic containing all instances of the Barcan scheme is strongly complete with respect to the class of quantifiable general frames validating it. We also provide a sufficient condition for the truth of all formulas, possibly with quantifiers, to be preserved under passing from a quantifiable general frame to its underlying Kripke frame. This is reminiscent of both the idea of elementary submodel in model theory and the persistence concepts in propositional modal logic. The key to this condition is the concept of finite generated diversity (Fritz 2023), and with it, we show that if Θ is a set of Sahlqvist formulas whose class of Kripke frames has finite generated diversity, then the smallest normal propositionally quantified modal logic containing Θ, Barcan, a formula stating the existence of world propositions, and a formula stating the definability of successor sets, is Kripke complete. As a special case, we have a simple finite axiomatization of the logic of Euclidean Kripke frames.

Keywords: Propositional quantifier, Sahlqvist formula, canonical frame, diversity, completeness

1 Introduction

Propositionally quantified modal logics (PQMLs henceforth) are modal logics augmented with propositional quantifiers, a special kind of quantifiers that can be intuitively understood as capturing the quantification implicit in English sentences such as "Everything Jane believes is false" and "It's likely that there's something I will never know". One can also understand the expression "For all the police knows, John is dead already" as "That John is dead already is compatible with everything the police knows" and see that propositional quantification is involved.

Normal propositional modal logic has been studied fruitfully in relation to the possible world-based Kripke frames. For propositional quantifiers, it is natural to consider them as quantifying over sets of possible worlds, since in Kripke frames, propositional variables are interpreted to sets of worlds, and propositional quantifiers bind these variables. This immediately gives PQMLs a second-order flavor, and indeed they are also known as second-order propositional modal logics when interpreted on Kripke frames. We may also impose a distinction between sets of worlds that count as propositions (possible value of propositional variables) and sets of worlds that do not count, thereby adding a *propositional domain* (also called the set of admissible sets) to Kripke frames and obtaining general frames. When a general frame validates the basic reasoning principles of propositional quantifiers, such as the instantiation reasoning from "Everything I believe is true" and "I believe that the Moon is made of cheese" to "The Moon is made of cheese", we call it a *quantifiable* frame.

While the theory of PQMLs based on Kripke frames and quantifiable frames has been studied since the early days of modal logic (see [15] for a nice survey), compared to other areas of modal logic, relatively little is known, and due to the second-order nature of PQMLs based on Kripke frames, many results are negative in the form of non-recursive axiomatizability. The early landmark paper is Fine's [10], in which among many other things, it is shown that the PQML consisting of formulas valid on the class of reflexive and transitive Kripke frames is not recursively axiomatizable while the PQML of the class of Kripke frames with a universal relation is decidable and can be axiomatized by simply adding to the modal logic $\mathsf{S5}$ the standard quantificational logic and an atomicity principle stating that there is always a world proposition that is itself true and entails every truth. After the mid-1990s, much work has been done in similar veins [21,20,1,23,6,24,3,22,11,30,8]. A recent breakthrough is by Fritz in [13], where he established many new results in the decidability and axiomatizability of PQMLs in a very general fashion. Prior to this work, it was not even known whether the PQML of Euclidean Kripke frames is decidable, and the decidability of PQML of the Kripke frames validating $\mathsf{KD45}$ was only established in [8].

In this paper, we focus on the question of when an axiomatically defined PQML is complete w.r.t. the Kripke/quantifiable frames it defines. For this to be possible, a necessary condition is that the PQML of the defined class is at least axiomatizable. Thus we build on the key idea behind the general decidability result in [13]: the *diversity* of Kripke frames. Given a Kripke frame $\mathbf{F}$, two worlds in it are called duplicates if the permutation switching them is an automorphism of $\mathbf{F}$. Being duplicates is an equivalence relation, and the diversity of $\mathbf{F}$ is the cardinality of the set of equivalence classes of this relation, while the *diversity* of a class C of Kripke frames is the supremum of the diversity of all *point-generated* subframes of frames in C. Fritz [13] shows that if a class C of Kripke frames is defined by a finite set Θ of formulas and has finite diversity, then the PQML of C is decidable. Decidability itself does not provide much information on whether there can be a simple and intuitive axiomatization or if a given axiomatically defined logic is complete, but for Kripke frames validating $\mathsf{KD45}$, it is shown [8] that we only need to add the quantification axioms and rules, the Barcan scheme $\mathtt{Bc}$, and the atomicity principle for completeness. Can this result be generalized? Our finding is that if Θ is a set of Sahlqvist formulas and the class $\mathsf{KFr}(\Theta)$ of Kripke frames it defines has diversity n, then the normal PQML axiomatized by Θ, the Barcan scheme $\mathtt{Bc}$, an atomicity principle $\mathtt{At}^n$, and an axiom $\mathtt{R}^n$ stating that successor sets are propositions, is sound and complete w.r.t. $\mathsf{KFr}(\Theta)$. The last two axioms are parametrized by n as they use iterated modalities to simulate the reflexive and transitive closure of the primitive modality. While we can replace the condition of Θ being Sahlqvist by suitable technical conditions, we cannot drop it completely; and while for $\mathsf{KD45}$ $\mathtt{R}^n$ is redundant, for $\mathsf{K5}$ it is not.

Our method is based on saturated (witnessed) maximally consistent sets and the saturated canonical general frame built from them. Fine [10] claims that this method can be used to show the completeness of $\mathsf{S5}$ with atomicity principle w.r.t. Kripke frames with a universal relation but provided only an extremely terse sketch. We will first show that if Λ is a normal PQML with Barcan, then its saturated canonical general frame is quantifiable and indeed validates Λ. A corollary is that any normal PQML containing $\mathtt{Bc}$ is sound and complete w.r.t. the class of quantifiable frames it defines. This basic fact may have been realized by multiple scholars before, but a formal proof for the fully general statement appears to be missing.

Starting with the saturated canonical general frame, we gradually turn it into a Kripke frame that validates the original logic and satisfies a given consistent formula. The strategy is as follows: (1) first take a point-generated general frame, then (2) keep only the worlds that are named by world propositions and obtain what we call the atomic general frame, and (3) finally show that we can expand the propositional domain of the atomic general frame to the full powerset without affecting the semantic value of any formula. Step (2) also happens in some completeness proofs of hybrid logics (see e.g. [28], Definition 5.2.6). The resulting atomic general frame has the special property of being discrete and tense, and Sahlqvist formulas are shown to be persistent over these general frames [29]. This is essential in showing that the final frame validates the original logic. Step (3) can also be utilized to show completeness for the monadic second-order theory of ω [25].

In Section 2 we review the basic definitions and provide a Tarski-Vaught-style test for the expansion of the propositional domain to the full powerset to preserve the value of all formulas with propositional quantifiers. In Section 3, we show that the saturated canonical general frame of any PQML containing $\mathtt{Bc}$ validates the logic. In Section 4, we show how finite diversity allows a quantifiable frame to pass the Tarski-Vaught test. In Section 5 we prove our main result. Finally, we conclude in Section 6.

2 Preliminaries

Definition 2.1 We fix a countably infinite set Prop of propositional variables and define the language $\mathcal{L}$ of PQMLs by the following grammar:

$$\varphi ::= p \mid \neg\varphi \mid (\varphi \vee \varphi) \mid \Diamond\varphi \mid \exists p\varphi$$

where $p \in \mathsf{Prop}$ and $\Diamond$ is the sole modality in this language. The other common connectives $\wedge, \to, \leftrightarrow, \Box$ and the universal quantifier $\forall p$ are defined as abbreviations as usual. Let $\mathsf{Fv}(\varphi)$ be the set of free variables of φ and $\mathcal{L}_{qf}$ the quantifier-free fragment of $\mathcal{L}$.

Definition 2.2 A *Kripke frame* is a pair $\mathbb{F} = (W, R)$ where W is non-empty and $R \subseteq W^2$. A *general frame* is a triple $\mathbf{F} = (W, R, B)$ where (W, R) is a Kripke frame and $B \subseteq \wp(W)$ is non-empty and closed under Boolean operations and $m^{\mathbf{F}}_{\Diamond}$, defined by $m^{\mathbf{F}}_{\Diamond}(X) = \{w \in W \mid \exists u \in X, wRu\}$. $R[w] = \{v \in W \mid wRv\}$ and $R[X] = \bigcup_{w \in X} R[w]$. When $B = \wp(W)$ we call $\mathbf{F}$ *full*.

A valuation v for $\mathbf{F}$ (or B) is a function from Prop to B. We define the semantic value $[\![\varphi]\!]^{\mathbf{F}}(v)$ of formulas φ relative to valuation v in the general frame $\mathbf{F} = (W, R, B)$ inductively by

$$[\![p]\!]^{\mathbf{F}}(v) = v(p) \quad [\![\neg\varphi]\!]^{\mathbf{F}}(v) = W \setminus [\![\varphi]\!]^{\mathbf{F}}(v) \quad [\![\varphi \vee \psi]\!]^{\mathbf{F}}(v) = [\![\varphi]\!]^{\mathbf{F}} \cup [\![\varphi]\!]^{\mathbf{F}}$$
$$[\![\Diamond\varphi]\!]^{\mathbf{F}}(v) = m^{\mathbf{F}}_{\Diamond}([\![\varphi]\!]^{\mathbf{F}}(v)) \quad [\![\exists p\varphi]\!]^{\mathbf{F}}(v) = \bigcup\{[\![\varphi]\!]^{\mathbf{F}}(v[X/p]) \mid X \in B\}.$$

Here $v[X/p]$ is the function that is identical to v except that $v[X/p](p) = X$. The intended meaning of $[\![\varphi]\!]^{\mathbf{F}}(v)$ is that it is the set of worlds where φ is true. A formula φ is *valid* on $\mathbf{F}$ when $[\![\varphi]\!]^{\mathbf{F}}(v) = W$ for all valuations v for $\mathbf{F}$.

Valuation and semantics for any Kripke frame $\mathbb{F} = (W, R)$ is defined the same as for $(W, R, \wp(W))$. That is, semantically a Kripke frame is equivalent to the full general frame based on it.

A *quantifiable frame* is a general frame $\mathbf{F} = (W, R, B)$ such that for any $v \in B^{\mathsf{Prop}}$ and any $\varphi \in \mathcal{L}$, $[\![\varphi]\!]^{\mathbf{F}}(v) \in B$. That is, B is 'closed under semantics'.

Notation: Given a Kripke frame (W, R) and two general frames $\mathbf{F} = (W, R, A)$ and $\mathbf{G} = (W, R, B)$ based on it, for any valuation $v \in (A \cap B)^{\mathsf{Prop}}$, it is clear that if $[\![\varphi]\!]^{\mathbf{F}}(v) \neq [\![\varphi]\!]^{\mathbf{G}}$, it is only because the difference between A and B. Hence, when it is clear which Kripke frame (W, R) is in discussion, we write $[\![\varphi]\!]^{(W,R,A)}$ simply as $[\![\varphi]\!]^{A}$ or even $[\![\varphi]\!]$ when φ is quantifier-free. Also, it is routine to show that $[\![\varphi]\!]^{\mathbf{F}}(v)$ only depends on $v|_{\mathsf{Fv}(\varphi)}$, the restriction of v to $\mathsf{Fv}(\varphi)$. Thus, for any partial function v from Prop to B such that $dom(v) \supseteq \mathsf{Fv}(\varphi)$, we take $[\![\varphi]\!]^{\mathbf{F}}(v)$ to be the unique element in $\{[\![\varphi]\!]^{\mathbf{F}}(v') \mid v \subseteq v' \in B^{\mathsf{Prop}}\}$.

Now we present the Tarski-Vaught-style test for expanding the propositional domain safely.

Definition 2.3 Given a Kripke frame (W, R) and $\varnothing \neq A, B \subseteq \wp(W)$, A is a Π-*invariant subdomain* of B if $A \subseteq B$ and for any $\varphi \in \mathcal{L}$ and $v \in A^{\mathsf{Prop}}$, $[\![\varphi]\!]^{A}(v) = [\![\varphi]\!]^{B}(v)$.

Lemma 2.4 *Given a Kripke frame (W, R) and $\varnothing \neq B \subseteq \wp(W)$, B is a Π-invariant subdomain of $\wp(W)$ iff for any $\varphi \in \mathcal{L}$, $p \in \mathsf{Fv}(\varphi)$, $v \in B^{\mathsf{Prop}}$, $w \in W$, and $X \in \wp(W)$, if $w \in [\![\varphi]\!]^{\wp(W)}(v[X/p])$ then there is $Y \in B$ with $w \in [\![\varphi]\!]^{\wp(W)}(v[Y/p])$.*

Proof. Left-to-Right: suppose B is a Π-invariant subdomain of $\wp(W)$ and $w \in [\![\varphi]\!]^{\wp(W)}(v[X/p])$. Then $w \in [\![\exists p\varphi]\!]^{\wp(W)}(v)$. Then by assumption, $w \in$

$[\![\exists p\varphi]\!]^B(v)$ and hence $w \in \bigcup\{[\![\varphi]\!]^B(v[Y/p]) \mid Y \in B\}$. So there is $Y \in B$ such that $w \in [\![\varphi]\!]^B(v[Y/p])$. Finally since B is a Π-invariant subdomain of $\wp(W)$, and $v[Y/p] \in B^{\mathsf{Prop}}$, $[\![\varphi]\!]^B(v[Y/p]) = [\![\varphi]\!]^{\wp(W)}(v[Y/p])$. Consequently there is $Y \in B$ such that $w \in [\![\varphi]\!]^{\wp(W)}(v[Y/p])$.

Right-to-Left: assume the stated criteria and use induction. Only the step for $\exists$ is non-trivial, where we need to show that $\bigcup\{[\![\varphi]\!]^{\wp(W)}(v[X/p]) \mid X \in \wp(W)\} = \bigcup\{[\![\varphi]\!]^B(v[Y/p]) \mid Y \in B\}$ with $v \in B^{\mathsf{Prop}}$. By IH, we only need to show that $\bigcup\{[\![\varphi]\!]^{\wp(W)}(v[X/p]) \mid X \in \wp(W)\} = \bigcup\{[\![\varphi]\!]^{\wp}(W)(v[Y/p]) \mid Y \in B\}$, and the right-to-left inclusion is trivial. For the other inclusion, take any $w \in \bigcup\{[\![\varphi]\!]^{\wp(W)}(v[X/p]) \mid X \in \wp(W)\}$ and use the criteria. □

Substitutions play an important role in our later proofs. Many authors define substitution only when it is free to do so, but we need the version that renames the bound variables when conflicts arise.

Definition 2.5 A *substitution* is a function $\sigma : \mathsf{Prop} \to \mathcal{L}$. Given a substitution σ, we extended it to $\mathcal{L}$ recursively so that that $\sigma(\neg\varphi) = \neg\sigma(\varphi)$, $\sigma(\Diamond\varphi) = \Diamond\sigma(\varphi)$, $\sigma(\varphi \vee \psi) = \sigma(\varphi) \vee \sigma(\psi)$, and $\sigma(\exists p\varphi) = \exists q\sigma^q_p(\varphi)$ where $q = p$ if $p \notin \bigcup_{r \in \mathsf{Fv}(\exists p\varphi)} \mathsf{Fv}(\sigma(r))$ and otherwise q is the first variable not used in $\exists p\varphi$ and any $\sigma(r)$ for $r \in \mathsf{Fv}(\exists p\varphi)$. Let ι be the identity substitution.

Thus, $\iota^\psi_p(\varphi)$ is the result of substituting in ψ for p in φ with the necessary renamings of bound variables. In particular, $\iota(\varphi) = \varphi$. Then, the standard substitution lemma connecting syntactic and semantic substitution is:

Lemma 2.6 *On any general frame* $\mathbf{F} = (W, R, B)$*, valuation* v *for* $\mathbf{F}$*, and substitution* σ*, define valuation* $\sigma \star v : p \mapsto [\![\sigma(p)]\!]^{\mathbf{F}}(v)$*. Then for any* $\varphi \in \mathcal{L}$*,* $[\![\varphi]\!]^{\mathbf{F}}(\sigma \star v) = [\![\sigma(\varphi)]\!]^{\mathbf{F}}(v)$*.*

Finally, we introduce logic. For the convenience of certain proofs, we opted for $\Diamond$ as the primitive modality. For this reason, the `Dual` axiom is necessary.

Definition 2.7 A *normal propositionally quantified modal logic* (NPQML) is a set $\Lambda \subseteq \mathcal{L}$ satisfying the following conditions:

- (Taut) all propositional tautologies are in Λ;
- axiom $\mathtt{K} = \Box(p \to q) \to (\Box p \to \Box q)$ and $\mathtt{Dual} = \Diamond p \leftrightarrow \neg\Box\neg p$ are in Λ;
- (UI) all instances of $\iota^\psi_p(\varphi) \to \exists p\varphi$ are in Λ;
- (Nec) if $\varphi \in \Lambda$, then $\Box\varphi \in \Lambda$;
- (MP) if $\varphi, (\varphi \to \psi) \in \Lambda$, then $\psi \in \Lambda$;
- (UG) if $\iota^q_p(\varphi) \to \psi \in \Lambda$ with $q \notin \mathsf{Fv}(\psi)$, then $\exists p\varphi \to \psi \in \Lambda$.

For any axioms or axiom schemes $A_1, A_2, \ldots, A_n$, we write $\mathsf{K}_\Pi A_1 A_2 \ldots A_n$ for the smallest NPQML containing all (instances) of all A_i's.

Recall that the famous *Barcan scheme* `Bc` is $\Diamond\exists p\varphi \to \exists p\Diamond\varphi$.

Fact 2.8 *For any class* C *of quantifiable frames, the set* Λ *of formulas valid on each member of* C *is a* NPQML *containing* `Bc`*.*

Definition 2.9 For any set $\Theta \subseteq \mathcal{L}$, let $\mathsf{KFr}(\Theta)$ be the class of Kripke frames validating all formulas in Θ and let $\Theta\pi+$ be the set of formulas valid on all members of $\mathsf{Kfr}(\Theta)$.

3 General completeness for quantifiable frames

This section introduces saturated canonical general frames for NPQMLs containing all instances of the Barcan scheme and shows that they validate the original logic and are automatically quantifiable. A consequence is the following:

Theorem 3.1 *Any* NPQML $\Lambda \supseteq \mathsf{K}_\Pi\mathsf{Bc}$ *is strongly complete w.r.t. the class of quantifiable frames validating* Λ.

Fix a NPQML$\Lambda \supseteq \mathsf{K}_\Pi\mathsf{Bc}$. To construct the saturated canonical general frame for Λ, we extend Prop to Prop^+ with countably infinitely many new variables and obtain the extended language $\mathcal{L}^+$. Semantics and logics for $\mathcal{L}^+$ are defined completely analogously. Let Λ^+ be the smallest NPQML in $\mathcal{L}^+$ extending Λ.

Definition 3.2 A set $\Gamma \subseteq \mathcal{L}$ is Λ-*consistent* if there is no finite $A \subseteq \Gamma$ s.t. $\neg(\bigwedge A) \in \Lambda$. A *maximally* Λ-*consistent set* (Λ-MCS) is a Λ-consistent set whose any proper extension in $\mathcal{L}$ is not Λ-consistent. Λ^+-consistency and Λ^+-MCSs are defined in the same way using Λ^+ and $\mathcal{L}^+$.

A Λ^+-MCS Γ is *saturated* if for any $\exists p\varphi \in \Gamma$, there is $q \in \mathsf{Prop}^+$ not occurring in $\exists p\varphi$ s.t. $\iota^q_p(\varphi) \in \Gamma$.

Now define the *saturated canonical general frame* $\mathbf{F}_\Lambda = (W, R, B)$ where

- W is the set of all saturated Λ^+-MCSs,
- wRv iff for all $\Box\varphi \in w$, $\varphi \in v$,
- $B = \{[\varphi] \mid \varphi \in \mathcal{L}^+\}$ where $[\varphi] = \{w \in W \mid \varphi \in w\}$.

The following two lemmas are completely analogous to their first-order modal logic counterparts. $\mathtt{Bc}$ is essential for Lemma 3.4 and a detailed proof of it can be found in Appendix A.

Lemma 3.3 *Any* Λ-*consistent* Γ *can be extended to a saturated* Λ^+-*MCS* Γ^+.

Lemma 3.4 *For any* $w \in W$, *if* $\Diamond\varphi \in w$, *then there is* $u \in R[w]$ *s.t.* $\varphi \in u$.

The truth lemma is replaced by a more general statement for all valuations arising from substitutions. This is a common idea in algebraic semantics.

Lemma 3.5 *For any substitution* σ *for* $\mathcal{L}^+$, *define its associated valuation* $[\sigma] : p \mapsto [\sigma(p)]$. *Then for any* $\varphi \in \mathcal{L}^+$ *and all* σ, $[\![\varphi]\!]^{\mathbf{F}_\Lambda}([\sigma]) = [\sigma(\varphi)]$.

Proof. By induction on φ. The cases for variables and negation go by

$$[\![p]\!]^{\mathbf{F}_\Lambda}([\sigma]) = [\sigma](p) = [\sigma(p)].$$
$$[\![\neg\varphi]\!]^{\mathbf{F}_\Lambda}([\sigma]) = W \setminus [\![\varphi]\!]^{\mathbf{F}_\Lambda}([\sigma]) = W \setminus [\sigma(\varphi)] = [\neg\sigma(\varphi)] = [\sigma(\neg\varphi)].$$

The case for disjunction is similar. For the modal case, note that Lemma 3.4 implies that for any φ, $[\Diamond\varphi] = m^{\mathbf{F}}_\Diamond([\varphi])$, so this is again easy.

For the quantifier case, recall that $\sigma(\exists p\varphi) = \exists q\sigma^q_p(\varphi)$ for a suitable q. Now observe that due to saturation and (UI),

$$[\exists q\sigma^q_p(\varphi)] \subseteq \bigcup\{[\iota^r_q\sigma^q_p(\varphi)] \mid r \in \mathsf{Prop}^+\} \subseteq \bigcup\{[\iota^\psi_q\sigma^q_p(\varphi)] \mid \psi \in \mathcal{L}^+\} \subseteq [\exists q\sigma^q_p(\varphi)].$$

Next, observe that $\iota^\psi_q\sigma^q_p(\varphi)$ and $\sigma^\psi_p(\varphi)$ are the same formula (the reasoning here is more delicate; one should discuss whether q is fresh or is just p). Also, $[\sigma^\psi_p] = [\sigma][[\psi]/p]$. Thus, with IH and recalling that $B = \{[\psi] \mid \psi \in \mathcal{L}^+\}$,

$$\begin{aligned}[\sigma(\exists p\varphi)] &= \bigcup\{[\sigma^\psi_p(\varphi)] \mid \psi \in \mathcal{L}^+\} = \bigcup\{[\![\varphi]\!]^{\mathbf{F}_\Lambda}([\sigma^\psi_p]) \mid \psi \in \mathcal{L}^+\}\\ &= \bigcup\{[\![\varphi]\!]^{\mathbf{F}_\Lambda}([\sigma][[\psi]/p]) \mid \psi \in \mathcal{L}^+\} = \bigcup\{[\![\varphi]\!]^{\mathbf{F}_\Lambda}([\sigma][X/p]) \mid X \in B\}\\ &= [\![\exists p\varphi]\!]^{\mathbf{F}_\Lambda}([\sigma]).\end{aligned}$$

□

Now, for any valuation v for $\mathbf{F}_\Lambda$, given how B is defined, there is a substitution σ such that $v = [\sigma]$. It follows that $\mathbf{F}_\Lambda$ is quantifiable. This also means that if $\varphi \in \Lambda \subseteq \Lambda^+$, then $[\varphi]^{\mathbf{F}_\Lambda}(v) = [\sigma(\varphi)] = W$ as by logic $\sigma(\varphi)$ is also in Λ^+. This means $\mathbf{F}_\Lambda$ validates Λ. Also, $[\varphi]^{\mathbf{F}_\Lambda}([\iota]) = [\varphi]$, which means under valuation $[\iota]$, each Λ^+-MCS is satisfied by itself. Hence Theorem 3.1 follows.

4 From finite diversity to Π-invariant subdomain

We first introduce the concepts of duplicates and diversity.

Definition 4.1 Given a Kripke frame $\mathbb{F} = (W, R)$, we say that worlds $w, u \in W$ are *duplicates* if the permutation (wu) of W that exchanges w and u is an automorphism of $\mathbb{F}$. Let Δ be this relation of being duplicates ($\mathbb{F}$'s duplication relation), which clearly is an equivalence relaiton on W, and then let W/Δ be the set of D's equivalence classes. The *diversity* of $\mathbb{F}$ is the cardinality of W/Δ. The *diversity* of a Kripke frame class is the supremum of the diversity of all point-generated subframes of the frames in that class (if exists).

The following lemma collects some easy but very useful properties of the duplicate classes and how they interact with R and $m_\Diamond$.

Lemma 4.2 *For any Kripke frame $\mathbb{F} = (W, R)$ and its duplication relation Δ,*

- *for any $D_1 \neq D_2 \in W/\Delta$, there is $w \in D_1$ and $u \in D_2$ s.t. wRu iff for all $w \in D_1$ and $u \in D_2$, wRu;*
- *for any $D \in W/\Delta$, the only possible configurations for $R|_D$: D^2, $\varnothing$, and when $|D| \geqslant 2$, $D^2 \setminus id_D$ and id_D (id_D is the identity relation on D).*

For convenience we define the binary relation R_Δ on W/Δ s.t. $D_1 R_\Delta D_2$ iff there is $w \in D_1$ and $u \in D_2$ s.t. wRu. Here we allow $D_1 = D_2$.

Now we discuss the possible ways $m^{\mathbf{F}}_\Diamond(X) \cap D$ is determined.

- *In case $R|_D = D^2$ or $R|_D = \varnothing$, clearly $m^{\mathbf{F}}_\Diamond(X) \cap D$ is either D or $\varnothing$, and it is D iff there is $D' \in R_\Delta[D]$ s.t. $|X \cap D'| \geqslant 1$.*
- *In case $R|_D = D \setminus id_D$ with $|D| \geqslant 2$, if there is $D' \in R_\Delta[D] \setminus \{D\}$ s.t. $|X \cap D'| \geqslant 1$, then $m^{\mathbf{F}}_\Diamond(X) \cap D = D$, otherwise,*

· *if* $|X \cap D| \geqslant 2$*, then also* $m^{\mathbf{F}}_{\Diamond}(X) \cap D = D$*,*
· *if* $|X \cap D| = 1$*, then* $m^{\mathbf{F}}_{\Diamond}(X) \cap D = D \setminus X$*, and*
· *if* $|X \cap D| = 0$*, then* $m^{\mathbf{F}}_{\Diamond}(X) \cap D = \varnothing$*.*

- *In case* $R|_D = id_D$ *with* $|D| \geqslant 2$*, if there is* $D' \in R_\Delta[D] \setminus \{D\}$ *s.t.* $|X \cap D'| \geqslant 1$*, then* $m^{\mathbf{F}}_{\Diamond}(X) \cap D = D$*, and otherwise,* $m^{\mathbf{F}}_{\Diamond}(X) \cap D = X$*.*

What follows is the core of the proof of our main result. We want to show that if the underlying Kripke frame of the quantifiable frame (W, R, B) has finite diversity and B contains all singleton and duplicate classes, then B is a Π-invariant subdomain of $\wp(W)$. The key idea is that whenever φ is true at w where at most one $p \in \mathsf{Fv}(\varphi)$ is evaluated to a set $X \subseteq W$ that is not necessarily in B, we can always swap the valuation of p to a $Y \in B$ while keeping φ true at w. For this to be true, we must establish that φ's truth at w is insensitive to certain changes in the valuation of p. For monadic second-order logic, this can be done with EF-game, but for modal logic, we cannot only focus on how φ's truth at w is insensitive to change since modality requires us to also consider the truth of φ at other worlds. We must take a more global perspective and strive to show that φ's 'meaning' is insensitive to certain changes. In the end, we arrive at a qualified quantifier-elimination: when restricted to a duplicate class D and relative to a valuation v, the 'meaning' of φ can be written as a Boolean formula $f_\varphi(v, D)$ using variables in $\mathsf{Fv}(\varphi)$, and when valuations u and v are close enough, $f_\varphi(v, D) = f_\varphi(u, D)$.

Definition 4.3 For any finite $\mathsf{p} \subseteq \mathsf{Prop}$, we write $\langle \mathsf{p} \rangle$ for the Boolean language generated from p. $at(\langle \mathsf{p} \rangle)$ is the finite set of all formulas $l_1 \wedge \cdots \wedge l_k$ where each l_i is either p_i or $\neg p_i$ and $p_1, \ldots, p_k$ list all elements in p. These formulas correspond to the atoms in the Lindenbaum algebra of $\langle \mathsf{p} \rangle$.

Definition 4.4 Given a Kripke frame (W, R) with duplicate relation Δ, for any finite $\mathsf{p} \subseteq \mathsf{Prop}$, $u, v \in \wp(W)^{\mathsf{p}}$, and $n \in \mathbb{N}$, $u \approx_n v$ if for all $D \in W/\Delta$ and $\zeta \in at(\langle \mathsf{p} \rangle)$, $|[\![\zeta]\!](u) \cap D| = |[\![\zeta]\!](v) \cap D|$ or both $|[\![\zeta]\!](u) \cap D|, |[\![\zeta]\!](v) \cap D| \geqslant 2^n$.

Lemma 4.5 *Let* $u, v \in \wp(W)^{\mathsf{p}}$ *for some finite* $\mathsf{p} \subseteq \mathsf{Prop}$*.*

(i) *If* $u \approx_n v$*, then not only for* $\zeta \in at(\langle \mathsf{p} \rangle)$*, for any* $\beta \in \langle \mathsf{p} \rangle$ *and* $D \in W/\Delta$*,* $|[\![\beta]\!](u) \cap D| = |[\![\beta]\!](v) \cap D|$ *or both* $|[\![\beta]\!](u) \cap D|, |[\![\beta]\!](v) \cap D| \geqslant 2^n$*.*

(ii) *If* $u \approx_n v$ *and* $p \notin \mathsf{p}$*, for any* $X \in \wp(W)$ *there is* $Y \in \wp(W)$ *s.t.* $u[X/p] \approx_{n-1} v[Y/p]$ *(since* $p \notin dom(u)$*,* $u[X/p] = u \cup \{(p, X)\}$*).*

A proof of this lemma can be found in Appendix B.

Lemma 4.6 *Given a Kripke frame* (W, R) *with duplicate relation* Δ*, for each* $\varphi \in \mathcal{L}$*, there is a function* $f_\varphi : (\wp(W)^{\mathsf{Fv}(\varphi)} \times W/\Delta) \to \langle \mathsf{Fv}(\varphi) \rangle$ *such that*

- *for any* $v \in \wp(W)^{\mathsf{Fv}(\varphi)}$ *and* $D \in W/\Delta$*,* $[\![\varphi]\!](v) \cap D = [\![f_\varphi(v, D)]\!](v) \cap D$*;*
- *for any* $u \approx_{qd(\varphi)+1} v \in \wp(W)^{\mathsf{Fv}(\varphi)}$*,* $f_\varphi(u, D) = f_\varphi(v, D)$ *for all* $D \in W/\Delta$*.*

In particular, recursively define f *as follows and it will witness the lemma: for the basic and Boolean cases:* $f_p(v, D) = p$*,* $f_{\neg\varphi}(v, D) = \neg f_\varphi(v, D)$*,*

$f_{\varphi\vee\psi}(v,D) = f_\varphi(v|_{\mathsf{Fv}(\varphi)}, D) \vee f_\psi(v|_{\mathsf{Fv}(\psi)}, D)$. For the modal case, we copy the analysis in Lemma 4.2 with $X = [\![f_\varphi(v,D)]\!](v)$:

- *In case $R|_D = D^2$ or $\varnothing$, if there is $D' \in R_\Delta[D]$ s.t. $|X \cap D'| \geqslant 1$ then $f_{\Diamond\varphi}(v,D) = \top$, otherwise $f_{\Diamond\varphi}(v,D) = \bot$.*
- *In case $R|_D = D \setminus id_D$ with $|D| \geqslant 2$, if there is $D' \in R_\Delta[D] \setminus \{D\}$ s.t. $|X \cap D'| \geqslant 1$, then $f_{\Diamond\varphi}(v,D) = \top$, otherwise,*
  - *if $|X \cap D| \geqslant 2$, then also $f_{\Diamond\varphi}(v,D) = \top$,*
  - *if $|X \cap D| = 1$, then $f_{\Diamond\varphi}(v,D) = \neg f_\varphi(v,D)$, and*
  - *if $|X \cap D| = 0$, then $f_{\Diamond\varphi}(v,D) = \bot$.*
- *In case $R|_D = id_D$ with $|D| \geqslant 2$, if there is $D' \in R_\Delta[D] \setminus \{D\}$ s.t. $|X \cap D'| \geqslant 1$, then $f_{\Diamond\varphi}(v,D) = \top$, and otherwise, $f_{\Diamond\varphi}(v,D) = f_\varphi(v,D)$.*

For the quantifier case let $f_{\exists p\varphi}(v,D)$ be the following:

$$\bigvee\{\zeta \in at(\langle \mathsf{Fv}(\exists p\varphi)\rangle) \mid [\![\zeta]\!](v) \cap \bigcup_{X \in \wp(W)} [\![f_\varphi(v[X/p], D)]\!](v[X/p]) \cap D \neq \varnothing\}.$$

Proof. We first show by induction that whenever $u \approx_{qd(\varphi)+1} v$, $f_\varphi(u,D) = f_\varphi(v,D)$. The base case and the inductive steps for Boolean connectives are trivial. For the modal case, suppose $u \approx_{qd(\Diamond\varphi)+1} v$. Then $u \approx_{qd(\varphi)+1} v$. Using IH, let $\beta = f_\varphi(u,D) = f_\varphi(v,D)$ and let $X = [\![\beta]\!](u), X' = [\![\beta]\!](v)$. Now at least $u \approx_1 v$, so for any $E \in W/\Delta$, either $|X \cap E| = |X' \cap E|$ or both $|X \cap E|$ and $|X' \cap E| \geqslant 2$. This means in the case analysis defining $f_{\Diamond\varphi}(u,D)$ and $f_{\Diamond\varphi}(v,D)$, the same case must be active, and $f_{\Diamond\varphi}(u,D) = f_{\Diamond\varphi}(v,D)$.

For the quantifier case, suppose $u \approx_{qd(\exists p\varphi)+1} v$ with $\mathsf{p} = \mathsf{Fv}(\exists p\varphi)$ and $u, v \in \wp(W)^{\mathsf{p}}$. Then $u \approx_{qd(\varphi)+2} v$. Now pick any $\zeta \in at(\langle \mathsf{p}\rangle)$ and suppose ζ is a disjunct of $f_{\exists p\varphi}(u,D)$. Then there is $X \in \wp(W)$ s.t. $[\![\zeta]\!](u) \cap [\![f_\varphi(u[X/p], D)]\!](u[X/p]) \cap D \neq \varnothing$. Since $p \notin \mathsf{p}$, $[\![\zeta]\!](u) = [\![\zeta]\!](u[X/p])$. So we have $[\![\zeta \wedge f_\varphi(u[X/p], D)]\!](u[X/p]) \cap D \neq \varnothing$. Now by Lemma 4.5, there is $Y \in \wp(W)$ s.t. $u[X/p] \approx_{qd(\varphi)+1} v[Y/p]$. So with IH, we can let $\beta = \zeta \wedge f_\varphi(u[X/p], D) = \zeta \wedge f_\varphi(v[Y/p], D)$ and now $[\![\beta]\!](u[X/p]) \cap D \neq \varnothing$. Then $[\![\beta]\!](v[Y/p]) \cap D \neq \varnothing$. This means ζ is also a disjunct of $f_{\exists p\varphi}(v,D)$. The above argument can be reversed, so $f_{\exists p\varphi}(u,D)$ and $f_{\exists p\varphi}(v,D)$ have the same disjuncts and thus are the same formula.

Now we show that $[\![\varphi]\!](v) \cap D = [\![f_\varphi(v,D)]\!](v) \cap D$. Again this is by induction and the non-quantifier cases are easy. For easy notation, let $\mathsf{p} = \mathsf{Fv}(\exists p\varphi)$ and $\beta_X = f_\varphi(v[X/p], D)$. By IH, $[\![\exists p\varphi]\!](v) \cap D = \bigcup_{X \in \wp(W)} [\![\beta_X]\!](v[X/p]) \cap D$. Given the definition of $f_{\exists p\varphi}(v,D)$ and that $\mathcal{C} := \{[\![\zeta]\!](v) \cap D \mid \zeta \in at(\langle \mathsf{p}\rangle)\}$ forms a partition of D, all we need to show is that $[\![\exists p\varphi]\!](v) \cap D$ is a union of cells in $\mathcal{C}$. For this, it suffices to show that for any $\zeta \in at(\langle \exists p\varphi\rangle)$ and $w_1, w_2 \in [\![\zeta]\!](v) \cap D$, if $w_1 \in \bigcup_{X \in \wp(W)} [\![\beta_X]\!](v[X/p]) \cap D$ then w_2 is also in $\bigcup_{X \in \wp(W)} [\![\beta_X]\!](v[X/p]) \cap D$. So suppose that $w_1, w_2 \in [\![\zeta]\!](v) \cap D$ for some $\zeta \in at(\langle p\rangle)$ and there is $X \in \wp(W)$ s.t. $w_1 \in [\![\beta_X]\!](v[X/p]) \cap D$. Recall that $(w_1 w_2)$ is the permutation of W that exchange w_1 and w_2. Let $Y = (w_1 w_2)[X]$. Since w_1, w_2 are both in $[\![\zeta]\!](v)$ and ζ is in $at(\langle \mathsf{p}\rangle)$, for any $q \in \mathsf{p}$, $v(q) = (w_1 w_2)[v(q)]$ as $w_1 \in v(q)$ iff

$w_2 \in v(q)$. Recall also that w_1, w_2 are in the same duplication class D. From all these, it is clear that $v[X/p] \approx_{qd(\varphi)+1} v[Y/p]$, since $(w_1w_2)[\cdot]$ commutes with all Boolean operations and thus for any $\gamma \in \langle \mathsf{p} \cup \{p\}\rangle$, $[\![\gamma]\!](v[Y/p]) \cap D = (w_1w_2)[[\![\gamma]\!](v[X/p]) \cap D]$, meaning also that they are of the same cardinality. Then $\beta_X = \beta_Y$. Recall that $w_1 \in [\![\beta_X]\!](v[X/p]) \cap D$. Apply (w_1w_2) to both sides and we have $w_2 \in [\![\beta_Y]\!](v[Y/p]) \cap D$. □

Theorem 4.7 *For any general frame (W, R, B) with (W, R) having finite diversity, B is a Π-invariant subdomain of $\wp(W)$ if for any $w \in W$ and $D \in W/\Delta$, $\{w\}, D \in B$. (Only B's closure under Boolean operations is used.)*

Proof. It suffice to show that for any $w \in W$, $\varphi \in \mathcal{L}$ with $\mathsf{p} = \mathsf{Fv}(\exists p\varphi)$ and $n = qd(\varphi) + 1$, $v \in B^{\mathsf{p}}$, and $X \in \wp(W)$, there is $Y \in B$ s.t. $v[X/p] \approx_n v[Y/p]$ and $w \in X$ iff $w \in Y$. Since if so, then by Lemma 4.6, $w \in [\![\varphi]\!](v[X/p]) \cap D$ iff $w \in [\![f_\varphi(v[X/p], D)]\!](v[X/p]) \cap D$ iff $w \in [\![f_\varphi(v[Y/p], D)]\!](v[X/p]) \cap D$ iff $w \in [\![\varphi]\!](v[Y/p]) \cap D$ and thereby by Lemma 2.4 we are done.

Notice that by assumption we have a finite partition $\mathcal{C} = \{[\![\zeta]\!](v) \cap D \mid \zeta \in at(\langle \mathsf{p}\rangle), D \in W/\Delta\}$ of W and each $C \in \mathcal{C}$ is in B by assumption. For $Y \in B$, it is enough to make sure that for all $C \in \mathcal{C}$, $C \cap Y$ or $C \setminus Y$ is finite. For $v[X/p] \approx_n v[Y/p]$, it is enough to make sure that for all $C \in \mathcal{C}$, $|C \cap Y| = |C \cap X|$ or $|C \cap Y|, |C \cap X| \geqslant 2^n$, and $|C \setminus Y| = |C \setminus X|$ or $|C \setminus Y|, |C \setminus X| \geqslant 2^n$. Thus, for each $C \in \mathcal{C}$, let $Y_C = C \cap X$ if either $C \cap X$ or $C \setminus X$ is finite, and otherwise when both are infinite, if $w \in C \cap X$, let Y_C be $C \setminus Z$ for some $Z \subseteq C \setminus X$ with $|Z| = 2^n$, and otherwise let Y_C be some subset of $C \cap X$ with $|Y_C| = 2^n$. Clearly $Y = \bigcup_{C \in \mathcal{C}} Y_C$ satisfies the requirements. □

5 General completeness with finite-diversity

We first define the extra axioms needed, whose validity over Kripke frames is easy to see using singleton sets and successor sets.

Definition 5.1 Recursively define $\Diamond^0\varphi = \Diamond^{\leqslant 0}\varphi = \varphi$ and $\Diamond^{n+1}\varphi = \Diamond\Diamond^n\varphi$ while $\Diamond^{\leqslant n+1}\varphi = \Diamond^{\leqslant n}\varphi \vee \Diamond^{n+1}\varphi$. Define $\Box^n$ and $\Box^{\leqslant n}$ dually. Then

- $\mathtt{Q}^n(\varphi) = \Diamond^{\leqslant n}\varphi \wedge \forall p(\Box^{\leqslant n}(\varphi \to p) \vee \Box^{\leqslant n}(\varphi \to \neg p))$ where p is the first variable not in $\mathsf{Fv}(\varphi)$;
- $\mathtt{At}^n = \forall q(\Diamond^{\leqslant n} q \to \exists p(\mathtt{Q}^{\leqslant n}(p) \wedge \Box^{\leqslant n}(p \to q)))$;
- $\mathtt{R}^n = \forall p \exists q(\Box^{\leqslant n}(p \to \Box q) \wedge \forall r(\Box^{\leqslant n}(p \to \Box r) \to \Box^{\leqslant n}(q \to r))$.

Theorem 5.2 *Let $\Theta \subseteq \mathcal{L}_{qf}$ be a set of Sahlqvist formulas s.t. the class $\mathsf{KFr}(\Theta)$ of Kripke frames validating Θ has diversity n. Then $\mathsf{K}_\Pi\Theta\mathtt{BcAt}^n\mathtt{R}^n$ is sound and strongly complete for $\mathsf{KFr}(\Theta)$.*

Fix a set Θ of Sahlqvist formulas with the generated diversity of $\mathsf{KFr}(\Theta)$ being n and let $\Lambda = \mathsf{K}_\Pi\Theta\mathtt{BcAt}^n\mathtt{R}^n$. We first establish a logical point:

Lemma 5.3 *$\Diamond^{n+1}p \to \Diamond^{\leqslant n}p$ is a theorem of Λ. Thus, denoting $\Diamond^{\leqslant n}$ by E and the dual $\Box^{\leqslant n}$ by A, Λ proves that A is an $\mathsf{S4}$ modality that commutes with $\forall$, and A works like the reflexive and transitive closure of $\Box$ in that for eaxmple (1) $\mathsf{E}\Diamond\varphi \to \mathsf{E}\varphi \in \Lambda$ and (2) for any $m \in \mathbb{N}$, $\Diamond^m\varphi \to \mathsf{E}\varphi \in \Lambda$.*

Proof. First we show that any point-generated Kripke frame $\mathbf{G} \in \mathsf{KFr}(\Theta)$ must also validate $\Diamond^{n+1}p \to \Diamond^{\leqslant n}p$. Suppose not, then we have some $wRx_1Rx_2\ldots Rx_nRu$, which is also a shortest path from w to u. This path is also present and shortest in the subframe $\mathbf{G}_w$ of $\mathbf{G}$ generated from w. Now note that $w, x_1, \ldots, x_n$ are pairwise non-duplicates within $\mathbf{G}_w$, since if there were a duplicate pair, then the path can be shortened. This contradicts that every point-generated frame in $\mathsf{KFr}(\Theta)$ has diversity at most n, as $\mathbf{G}_w$ clearly is also in $\mathsf{KFr}(\Theta)$. Since Θ is Kripke complete, $\Diamond^{n+1}p \to \Diamond^{\leqslant n}p$ is in Λ. The remaining claims follow easily from basic normal modal reasoning and $\mathtt{Bc}$. □

In the following we continue using A for $\Box^{\leqslant n}$ and E for $\Diamond^{\leqslant n}$ and drop the superscripts on $\mathtt{Q}^n$, $\mathtt{At}^n$, and $\mathtt{R}^n$.

Now we start with the canonical saturated general frame $\mathbf{F}_\Lambda = (W, R, B)$. Recall that this involves expanding the language to $\mathcal{L}^+$ built from variables in Prop^+ and extending Λ conservatively to Λ^+. Lemma 5.3 transfer to Λ^+ with no problem. For Theorem 5.2, clearly it is enough to show that every $w \in W$ can be satisfied in a Kripke frame validating Θ. Thus fix an arbitrary $a \in W$ and consider the general frame $\mathbf{F}_a = (W_a, R_a, B_a)$ generated from a, defined as follows:

- W_a is $R^*[a]$ where R^* is the reflexive and transitive closure of R;
- $R_a = R \cap (W_a \times W_a)$;
- $B_a = \{X \cap W_a \mid X \in B\}$; we write $[\varphi]_a$ for $[\varphi] \cap W_a$, and with this notation, $B_a = \{[\varphi]_a \mid \varphi \in \mathcal{L}^+\}$.

We show that A and E work as universal and existential modalities at a and $\mathtt{Q}$ works as intended.

Lemma 5.4 *For any $\varphi \in \mathcal{L}^+$, $\mathsf{A}\varphi \in a$ iff $[\varphi]_a = W_a$, and similarly $\mathsf{E}\varphi \in a$ iff $[\varphi]_a$ is non-empty. Also, $\mathtt{Q}(\varphi) \in a$ iff $[\varphi]_a$ is a singleton.*

Proof. For the first part, we just prove $\mathsf{E}\varphi \in a$ iff $[\varphi]_a \neq \varnothing$. Since a is an MCS and together with Lemma 5.3, $\mathsf{E}\varphi \in a$ iff there is m s.t. $\Diamond^m\varphi \in a$. By standard reasoning in canonical models, i.e., repeated use of Lemma 3.4, this is true iff there is $u \in W_a$ s.t. $\varphi \in u$.

Now for $\mathtt{Q}(\varphi)$, again since a is a saturated MCS, $\mathtt{Q}(\varphi) \in a$ iff (1) $\mathsf{E}\varphi \in a$ and (2) for any $\psi \in \mathcal{L}^+$, $\mathsf{A}(\varphi \to \psi) \vee \mathsf{A}(\varphi \to \neg\psi) \in a$. (The second point uses saturation.) Using the first part of this lemma, (1) translates to $[\varphi]_a \neq \varnothing$, and (2) translates to that for any $\psi \in \mathcal{L}^+$, $[\varphi]_a \subseteq [\psi]_a$ or $[\varphi]_a \subseteq (W_a \setminus [\psi]_a)$. If $[\varphi]_a$ is a singleton, these two points are clearly true. Conversely, if $[\varphi]_a$ is empty, (1) is clearly false. If instead there are distinct $x, y \in [\varphi]_a$, then there must be a formula $\psi \in \mathcal{L}^+$ s.t. $\psi \in x$ but $\psi \notin y$, making (2) false. □

This means the atoms of B_a (as a Boolean algebra under set inclusion) are precisely $\{[\varphi]_a \mid \mathtt{Q}(\varphi) \in a\}$. We focus on these atoms and define the *atomic subframe* $\mathbf{F}_a^{at}$ of $\mathbf{F}_a$ as $(W_a^{at}, R_a^{at}, B_a^{at})$ where

- $W_a^{at} = \{w \in W_a \mid \{w\} \in B_a\}$;
- $R_a^{at} = R_a \cap (W_a^{at} \times W_a^{at})$;

- $B_a^{at} = \{X \cap W_a^{at} \mid X \in B_a\}$.

We write $[\varphi]_a^{at} = [\varphi]_a \cap W_a^{at}$. Then $B_a^{at} = \{[\varphi]_a^{at} \mid \varphi \in \mathcal{L}^+\}$. A key property of $\mathbf{F}_a^{at}$ is that every world in it is named by a formula given how W_a^{at} is defined. For each $w \in W_a^{at}$, we fix a formula χ_w s.t. $\{w\} = [\chi_w]_a$.

We want to immediately make sure that $a \in W_a^{at}$.

Lemma 5.5 *The singleton $\{a\}$ is in B_a and thus $a \in W_a^{at}$.*

Proof. By a formal derivation, $\exists p(p \wedge \mathtt{Q}(p))$ is in Λ. Indeed, with $\mathsf{S4}$ normal modal reasoning and $\mathtt{Bc}$, we can derive $\mathtt{Q}(p) \to \mathsf{A}(p \to \mathtt{Q}(p))$ in Λ^+. The main steps include

- $(\mathsf{A}(p \to q) \vee \mathsf{A}(p \to \neg q)) \to \mathsf{A}(\mathsf{A}(p \to q) \vee \mathsf{A}(p \to \neg q))$
- $\forall q(\mathsf{A}(p \to q) \vee \mathsf{A}(p \to \neg q)) \to \mathsf{A}\forall q(\mathsf{A}(p \to q) \vee \mathsf{A}(p \to \neg q))$
- $\mathsf{A}(p \to \mathsf{E}p)$ and then $\forall q(\mathsf{A}(p \to q) \vee \mathsf{A}(p \to \neg q)) \to \mathsf{A}(p \to \mathtt{Q}(p))$.

Now suppose $\neg\exists p(p \wedge \mathtt{Q}(p))$. Then we derive $\mathsf{E}\forall p(p \to \neg\mathtt{Q}(p))$. With $\mathtt{At}$, we derive $\exists p(\mathtt{Q}(p) \wedge \mathsf{A}(p \to \forall p(p \to \neg\mathtt{Q}(p))))$. A contradiction follows. From $\mathsf{A}(p \to \forall p(\mathtt{Q}(p) \to \neg p))$ we have $\mathsf{A}(p \to \neg\mathtt{Q}(p))$. And recall $\mathtt{Q}(p) \to \mathsf{A}(p \to \mathtt{Q}(p))$ is in Λ^+. This means we derive $\mathsf{A}\neg p$, which contradicts the $\mathsf{E}p$ part in $\mathtt{Q}(p)$.

Thus $\exists p(p \wedge \mathtt{Q}(p)) \in a$, and since a is saturated, there is $r \in \mathsf{Prop}^+$ s.t. $r \wedge \mathtt{Q}(r) \in a$. Then $[r]_a$ must be $\{a\}$ and $\{a\} \in B_a$. $\square$

From this, we show that $\mathbf{F}_a^{at}$ behaves as a canonical general frame and thus is quantifiable:

Lemma 5.6
- *For any $w \in W_a^{at}$ and $\varphi \in \mathcal{L}^+$, $\varphi \in w$ iff $\mathsf{A}(\chi_w \to \varphi) \in a$ iff $\mathsf{E}(\chi_w \wedge \varphi) \in a$. Also, for any $w, u \in W_a^{at}$, $wR_a^{at}u$ iff $\mathsf{E}(\chi_w \wedge \Diamond\chi_u) \in a$.*
- *For any $\Diamond\varphi \in w \in W_a^{at}$, there is $u \in R_a^{at}[w]$ s.t. $\varphi \in u$.*
- *For any substitution σ, define the associated valuation $[\sigma]_a^{at}$ for $\mathbf{F}_a^{at}$ by $[\sigma]_a^{at}(p) = [\sigma(p)]_a^{at}$. Then $[\![\varphi]\!]^{\mathbf{F}_a^{at}}([\sigma]_a^{at}) = [\sigma(\varphi)]_a^{at}$.*

Proof. By definition, $[\chi_w]_a = \{w\}$ and $[\chi_u]_a = \{u\}$. As we have shown, $\mathsf{E}(\chi_w \wedge \Diamond\chi_u) \in a$ iff $[\chi_w \wedge \Diamond\chi_u]_a$ is non-empty, iff $\Diamond\chi_u \in w$, and iff $wR_a^{at}u$.

Now suppose $\Diamond\varphi \in w \in W_a^{at}$. First, apply $\mathtt{R}$ to χ_w and using saturation at a, we have a formula (indeed a variable) which we denote by $\chi_{R(w)}$ s.t. $\mathsf{A}(\chi_w \to \Box\chi_{R(w)}) \wedge \forall r(\mathsf{A}(\chi_w \to \Box r) \to \mathsf{A}(\chi_{R(w)} \to r))$ is in a. By pluging in $\neg r$ for r and contraposing, we have $\forall r(\mathsf{E}(\chi_{R(w)} \wedge r) \to \mathsf{E}(\chi_w \wedge \Diamond r)) \in a$. Also, from $\mathsf{A}(\chi_w \to \Box\chi_{R(w)}) \in a$ and $\Diamond\varphi \in w$, $\mathsf{E}(\chi_w \wedge \Diamond(\chi_{R(w)} \wedge \varphi)) \in a$. This means $\mathsf{E}\Diamond(\chi_{R(w)} \wedge \varphi)$ and thus $\mathsf{E}(\chi_{R(w)} \wedge \varphi) \in a$. Then by $\mathtt{At}$ (or the atomicity of B_a), there is $u \in W_a^{at}$ s.t. φ and $\chi_{R(w)} \in u$. Then $\mathsf{E}(\chi_{R(w)} \wedge \chi_u) \in a$, and thus $\mathsf{E}(\chi_w \wedge \Diamond\chi_u) \in a$, which means $u \in R_a^{at}(w)$.

For the last part, recall that for the original canonical saturated frame $\mathbf{F}_\Lambda$, Lemma 3.5 applies: for any φ and substitution σ for $\mathcal{L}^+$, $[\varphi]^{\mathbf{F}_\Lambda}([\sigma]) = [\sigma(\varphi)]$. $\mathbf{F}_a^{at}$ is obtained by restricting $\mathbf{F}_\Lambda$ to W_a^{at}. Thus, it is enough to show that for any valuation v for $\mathbf{F}_\Lambda$, writing $v|_a^{at}$ for the restricted valuation defined by $v|_a^{at}(p) = v(p) \cap W_a^{at}$, $[\varphi]^{\mathbf{F}_a^{at}}(v|_a^{at}) = [\varphi]^{\mathbf{F}_\Lambda}(v) \cap W_a^{at}$. Using induction on φ, the base and the Boolean cases are trivial, as the operation of relative negation and intersection commutes intersecting with W_a^{at}. For the modal case, we need $m_\Diamond^{\mathbf{F}_a^{at}}(X \cap W_a^{at}) = m_\Diamond^{\mathbf{F}_\Lambda}(X) \cap W_a^{at}$ where X is assumed to be in B, as $[\varphi]^{\mathbf{F}_\Lambda}(v)$ is always in B. The left-to-right inclusion is trivial since $\mathbf{F}_a^{at}$ is a restriction of $\mathbf{F}_\Lambda$. For the right-to-left inclusion, first write X as $[\psi]$ for some $\psi \in \mathcal{L}^+$ and use the second bullet point of this lemma. The quantifier case is not much different from the case for disjunction, using only the distribution of intersection over arbitrary union and that $B_a^{at} = \{X \cap W_a^{at} \mid X \in B\}$. □

Now we start to show that B_a^{at} has what it takes to be a Π-invariant subdomain of $\wp(W_a^{at})$ over the underlying Kripke frame $\mathbb{F}_a^{at} = (W_a^{at}, R_a^{at})$.

Lemma 5.7
- *For any $w \in W_a^{at}$, $\{w\} \in B_a^{at}$.*
- *For any $X \in B_a^{at}$, $R_a^{at}[X] \in B_a^{at}$.*
- *$\mathbf{F}_a^{at}$ is point-generated from a and $\mathbb{F}_a^{at} \vDash \Theta$. Thus it has diversity $k \leqslant n$.*
- *Let Δ be the duplicate relation of $\mathbb{F}_a^{at}$. Then each $D \in W_a^{at}/\Delta$ is in B_a^{at}.*

Proof. The first bullet point is trivial. For the second bullet point, pick any $X \in B_a^{at}$. Then we have some φ s.t. $X = [\varphi]_a^{at}$. Reasoning in the saturated MCS a and apply $\mathtt{R}$ to φ, we obtain a ψ s.t. $\mathsf{A}(\varphi \to \Box\psi) \in a$ and for any $\gamma \in \mathcal{L}^+$, $\mathsf{E}(\psi \wedge \gamma) \to \mathsf{E}(\varphi \wedge \Diamond\gamma) \in a$. Now for any $w \in X$, $\varphi \in w \in W_a^{at}$. Thus $\mathsf{E}(\chi_w \wedge \varphi) \in a$. Together with $\mathsf{A}(\varphi \to \Box\psi)$, $\mathsf{E}(\chi_w \wedge \Box\psi) \in a$, meaning $\Box\psi \in w$. Thus $R[w] \subseteq [\psi]$ and hence $R_a^{at}[w] \subseteq [\psi]_a^{at}$. Since w is chosen arbitrarily from X, $R_a^{at}[X] \subseteq [\psi]_a^{at}$. On the other hand, suppose $u \in [\psi]_a^{at}$. Then $\mathsf{E}(\psi \wedge \chi_u) \in a$. Then $\mathsf{E}(\varphi \wedge \Diamond\chi_u) \in a$ and thus there is $w \in W_a^{at}$ s.t. φ and $\Diamond\chi_u$ are in w. This means $w \in X$ and $wR_a^{at}u$. So in sum, $[\psi]_a^{at} \subseteq R_a^{at}[X]$.

For the third bullet point, we need to first show that every $w \in W_a^{at}$ is reachable from a within $\mathbf{F}_a^{at}$. If $w \in W_a^{at}$, then at least $\mathsf{E}\chi_w \in a$. This means for some $m \leqslant n$, $\Diamond^m\chi_w \in a$. By repeated use of the second bullet point of Lemma 5.6, there is indeed a path from a to w inside $\mathbf{F}_a^{at}$. Now, by the third bullet point of Lemma 5.6, $\mathbf{F}_a^{at} \vDash \Theta$. For this to transfer to $\mathbb{F}_a^{at}$, we rely on the assumption that Θ consists of Sahlqvist formulas, which are $\mathcal{AT}$-persistent in the sense that if they are valid on a general frame whose set of admissible sets contains all singletons (the atomic/$\mathcal{A}$ part) and is closed under taking successor set (the tense/$\mathcal{T}$ part), then they are also valid on the underlying Kripke frame. That Shalqvist formulas are $\mathcal{AT}$-persistent has been observed for example in [29]. The idea is that for any $\varphi \in \Theta$, if $\mathbb{F}_a^{at} \nvDash \varphi$, then there is a falsifying valuation that only uses sets in B_a^{at} so that it is also a valuation for $\mathbf{F}_a^{at}$, contradicting that $\mathbf{F}_a^{at} \vDash \Theta$. This special valuation is obtained by the standard minimal valuation technique for Sahlqvist formulas and note that in minimal valuations, only finite unions of sets of the form $(R_a^{at})^m[\{w\}]$ are used, which are in B_a^{at} by the assumed closure properties.

Now that $\mathbb{F}_a^{at}$ has diversity $k \leqslant n$, there are also $b_1, b_2, \ldots, b_k \in W_a^{at}$ each representing a duplicate class. For each i, we show that the duplicate class D that b_i is in is in B_a^{at}. Now if $D = \{b_i\}$ then we have shown that it is in B_a^{at}. So assume that there is $c \neq b_i$ in D. Observe that for any $w \in W_a^{at} \setminus \{b_1, \ldots, b_k\}$, $w \in D$ iff w and c are duplicates, and iff the following are true:

- for any j, $wR_a^{at}b_j$ iff $cR_a^{at}b_j$;
- for any j, $b_jR_a^{at}w$ iff $b_jR_a^{at}c$;
- w is reflexive iff c is reflexive;
- $wR_a^{at}c$ iff $cR_a^{at}w$.

The above conditions are all expressible in B_a^{at} using singletons, the $R_a^{at}[\cdot]$ operation, the $m_\Diamond^{\mathbb{F}_a^{at}}$ operation, and also the set of reflexive points in $\mathbb{F}_a^{at}$ defined by sentence $\forall p(\Box p \to p)$. In fact, in the original canonical saturated general frame $\mathbb{F}_\Lambda$, $[\forall p(\Box p \to p)]$ is already the set of all reflexive worlds since any two worlds are separated by a proposition in B. Indeed, the set

$$\begin{aligned}\{m_\Diamond^{\mathbb{F}_a^{at}}(\{b_j\}) \mid cR_a^{at}b_j\} \cup \{W_a^{at} \setminus m_\Diamond^{\mathbb{F}_a^{at}}(\{b_j\}) \mid \text{not } cR_a^{at}b_j\} \cup \\ \{R_a^{at}[b_j] \mid b_jR_a^{at}c\} \cup \{W_a^{at} \setminus R_a^{at}(b_j) \mid \text{not } b_jR_a^{at}c\} \cup \\ \{[\forall p(\Box p \to p)]_a^{at} \mid cR_a^{at}c\} \cup \{W_a^{at} \setminus [\forall p(\Box p \to p)]_a^{at} \mid \text{not } cR_a^{at}c\} \cup \\ \{(m_\Diamond^{\mathbb{F}_a^{at}}(\{c\}) \cap R_a^{at}[c]) \vee ((W_a^{at} \setminus m_\Diamond^{\mathbb{F}_a^{at}}(\{c\})) \cap (W_a^{at} \setminus R_a^{at}[c]))\}\end{aligned}$$

contains all the required conditions, the intersection of which we denote by X. Then $(X \setminus \{b_1, \ldots, b_k\}) \cup \{b_i\}$ is the duplicate calss D that b_i is in. □

Putting pieces together, for any Λ^+-MCS Σ, it is satisfied on $\mathbf{F}_a^{at}$ by itself under valuation $[\iota]_a^{at}$ by Lemma 5.6. By Lemma 5.7 and Theorem 4.7, B_a^{at} is a Π-invariant subdomain of $\wp(W_a^{at})$. By definition of Π-invariant subdomain and $[\iota]_a^{at} \in (B_a^{at})^{\mathsf{Prop}}$, Σ is satisfied on $\mathbb{F}_a^{at}$ by itself under valuation $[\iota]_a^{at}$. Finally, by the third bullet point of Lemma 5.7, $\mathbb{F}_a^{at}$ is a frame validating Θ. This completes the proof of theorem 5.2.

It is easy to observe that $\mathsf{KFr}(5)$ has diversity 3 and validates $\Diamond^3 p \to \Diamond^2 p$. Thus, the NPQML of Euclidean Kripke frames can be axiomatized as $\mathsf{K}_\Pi\mathtt{5BcAt}^2\mathtt{R}^2$. However, as we show in the appendix C, $\mathsf{K}_\Pi\mathtt{5BcAt}^2$ is not complete as it does not derive $\mathtt{R}^2$.

The requirement of Θ consisting of Sahlqvist formulas cannot be dispensed with altogether either. Consider the axioms $\mathtt{T}$: $p \to \Diamond p$, $\mathtt{M}$: $\Diamond\Box\neg p \vee \Diamond\Box p$, $\mathtt{E}$: $\Diamond(\Diamond p \wedge \Box q) \to \Box(\Diamond p \vee \Box q)$, and $\mathtt{Q}$: $(\Diamond p \wedge \Box(p \to \Box p)) \to p$ (we reuse the letter $\mathtt{Q}$) used in [27]. It is not hard to show that the only Kripke frames validating $\mathtt{TMEQ}$ are those with the identity accessibility relation, and thus $\mathsf{KFr}(\mathtt{TMEQ})$ has diversity 1. However, $\mathsf{K}_\Pi\mathtt{TMEQBcAt}^n\mathtt{R}^n$ is not the NPQML of $\mathsf{KFr}(\mathtt{TMEQ})$, as it does not derive $p \leftrightarrow \Box p$ that is valid on Kripke frames with the identity accessibility relation. Again, we show this in the appendix.

By observing where we used the Sahlqvist condition on Θ, we note a different way of stating our main result.

Definition 5.8 For any formula $\varphi \in \mathcal{L}$, it is $\mathcal{ATQ}$-persistent if for any quantifiable general frame (W, R, B) such that φ valid on it, all singleton subsets of W is in B, and B is closed under $R[\cdot]$, then φ is valid on the underlying (W, R).

Definition 5.9 For any $\Theta \subseteq \mathcal{L}$, it is Kripke $\mathcal{L}_{qf}$-complete if for any $\varphi \in \mathcal{L}_{qf}$ that is valid on all Kripke frames in $\mathsf{KFr}(\Theta)$, φ is derivable in $\mathsf{K}_\Pi\Theta$.

Corollary 5.10 *Let $\Theta \subseteq \mathcal{L}$ be a set of $\mathcal{ATQ}$-persistent formulas such that* $\mathsf{KFr}(\Theta)$ *has diversity n and Θ is Kripke $\mathcal{L}_{qf}$-complete. Then* $\mathsf{K}_{\Pi}\Theta\mathtt{BcAt}^n\mathtt{R}^n$ *is the NPQML of* $\mathsf{KFr}(\Theta)$*.*

6 Conclusion

Our first result that all NPQMLs containing `Bc` is complete for the class of quantifiable frames it defines is fairly standard and expected. We believe it would be instructive to rewrite this proof in terms of Lindenbaum algebra and duality theory. This may help us generalize this result, especially to all quantifiable *neighborhood* frames. Of course, `Bc` will be dropped from the logic, and similar work has been done for first-order modal logic [2]. One may also consider the alternative semantics in [16] where $\forall p\varphi$ is true at w iff there is a proposition X that contains w and entails all propositions expressible by φ as we vary the proposition denoted by p. Given the quantifiability requirement ($\forall p\varphi$ itself must express a proposition), there should be an easy general completeness result from duality theory.

For our second result, there are two natural ideas to generalize. The first is dropping $\mathtt{At}^n$ and consider completeness w.r.t. algebraic semantics based on complete and completely multiplicative modal algebras. It should be noted that, as is clear in our proof, axiom $\mathtt{At}^n$ corresponds to the existence of 'world propositions' that can later be interpreted as possible worlds, and the world propositions serve as the names of the possible worlds. Hybrid logics use world propositions in a much more direct way by taking them as a primitive syntactical category, namely the nominals, and a recent work [4] has considered propositionally quantified hybrid modal logic. As is mentioned in that paper, Arthur Prior is a strong proponent of both. However, the idea of there being no maximally specified possible worlds but only partial states [19,26,18,9] is also worth investigating in this context (though PQML together with plural quantifiers are used to argue for there being world propositions [14]), and algebraic semantics allowing atomless elements in the algebras is a natural way to model this. Previous work in this line include [17,7,8].

The other direction for generalization is dropping the finite diversity condition in some way. Indeed, frames with finite diversity are so thoroughly finite that it is likely with only finitely many propositional variables, only finitely many inequivalent formulas can be written within a given quantification rank, and the completeness is essentially due to a reduction to a quantifier-free normal form. We see that there is at least one promising way of relaxing the finite diversity condition: requiring only finite diversity for each point-generated frame of finite depth.

Finally, we mention a broader question: can the theory of PQMLs inform the theory of modal μ-calculus [5] or vice versa, especially over completeness questions, noting that the μ operator is also a kind of propositional quantifier? For example, the recent work [12] utilized the fixpoint construction in NPQML, and we believe more needs to be done.

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Appendix

A: Proof of Lemma 3.4

The proof is a standard exercise in modal logic with quantifiers.

Proof. Let $\varphi_1, \varphi_2 \dots$ be an enumeration of existential formulas in $\mathcal{L}^+$. Recursively we find a sequence of formula $\psi_0, \psi_1, \dots$ such that:

- $\psi_0 = \varphi$
- For some propositional variable $q \in \mathcal{L}^+$, $\psi_{n+1} = \psi_n \wedge (\exists p\gamma \to \iota_p^q(\gamma))$ where $\exists p\gamma = \varphi_n$.
- $\{\delta \mid \Box\delta \in w\} \cup \{\psi_{n+1}\}$ is consistent.

Assume such a sequence has been found, then $\{\delta \mid \Box\delta \in w\} \cup \{\psi_0, \psi_1, \dots\}$ is a consistent set. It is easy to check any Λ^+-MCS expansion v of it is saturated, and moreover wRv and $\varphi \in v$.

Now we show that such a sequence can always be found.

The base case is a standard existence lemma argument. Given ψ_n is found, we show that ψ_{n+1} can always be found. Suppose for contradiction that for all propositional variable $q \in \mathsf{Prop}^+$, $\{\delta \mid \Box\delta \in w\} \cup \{\psi_n \wedge (\exists p\gamma \to \iota_p^q(\gamma)\}$ is inconsistent.

Then for any $q \in \mathsf{Prop}^+$, there will be $\delta_1 \dots \delta_m \in \{\delta \mid \Box\delta \in w\}$ s.t.

$$(\delta_1 \wedge \cdots \wedge \delta_m) \to (\psi_n \to \neg(\exists p\gamma \to \iota_p^q(\gamma))) \in \Lambda^+ \tag{1}$$

By necessitation and K,

$$(\Box\delta_1 \wedge \cdots \wedge \Box\delta_m) \to \Box(\psi_n \to \neg(\exists p\gamma \to \iota_p^q(\gamma))) \in \Lambda^+ \tag{2}$$

Since $\Box\delta_i \in w$ and w is maximally consistent, $\Box(\psi_n \to \neg(\exists p\gamma \to \iota_p^q(\gamma))) \in w$. Notice that q can be any propositional variable in Prop^+. Pick some $r \in \mathsf{Prop}^+$ not in ψ_n and γ, and observe that by saturation of w, $\forall r\Box(\psi_n \to \neg(\exists p\gamma \to \iota_p^r(\gamma))) \in w$. This is because if not, then $\exists r\neg\Box(\psi_n \to \neg(\exists p\gamma \to \iota_p^r(\gamma))) \in w$, and since w is saturated, there is $q' \in \mathsf{Prop}^+$ s.t. $\iota_r^{q'}(\neg\Box(\psi_n \to$

$\neg(\exists p\gamma \to \iota_p^r(\gamma)))) \in w$, which means $\neg\Box(\psi_n \to \neg(\exists p\gamma \to \iota_p^{q'}(\gamma))) \in w$ since r is not in ψ_n and $\exists p\gamma$. But we have established that for any $q \in \mathsf{Prop}^+$, $\Box(\psi_n \to \neg(\exists p\gamma \to \iota_p^q(\gamma))) \in w$. Then

- $\forall r\Box(\psi_n \to \neg(\exists p\gamma \to \iota_p^r(\gamma))) \in w$ (shown above);
- $\Box\forall r(\psi_n \to \neg(\exists p\gamma \to \iota_p^r(\gamma))) \in w$ (By $\mathtt{BC}$);
- $\Box(\psi_n \to \forall r\neg(\exists p\gamma \to \iota_p^r(\gamma))) \in w$ (r is not free in ψ_n);
- $\Box\neg\psi_n \in w$ ($\exists r(\exists p\gamma \to \iota_p^r(\gamma)) \in \mathsf{K}_\Pi^+$).

Thus, $\neg\psi_n \in \{\delta \mid \Box\delta \in w\} \cup \{\psi_n\}$, a contradiction to its assumed consistency. Therefore, ψ_{n+1} can always be found that satisfies the required conditions. $\Box$

B: Proof of Lemma 4.5

Proof. The first part is easy since every $[\![\beta]\!](u)$ is the union of some $[\![\zeta]\!](u)$ where $\zeta \in at(\langle \mathsf{p}\rangle)$.

The second part: notice $\mathcal{C} = \{D \cap [\![\zeta]\!](u) \mid D \in W/\Delta, \zeta \in at(\langle \mathsf{p}\rangle)\}$ is a partition of W. For each $C \in \mathcal{C}$, let $Y_C \subseteq C$ satisfy:

$$\begin{cases} |Y_C| = |C \cap X| & \text{if } |C \cap X| < 2^{n-1} \\ |C \setminus Y_C| = |C \setminus X| & \text{if } |C \setminus X| < 2^{n-1} \leq |C \cap X| \\ |Y_C| = 2^{n-1} & \text{if both } |C \cap X|, |C \setminus X| > 2^{n-1} \end{cases}$$

Given $u \approx_n v$, such Y_C can always be found and the specific choice of Y_C is irrelevant. Take $Y = \bigcup_{C\in\mathcal{C}} Y_C$, then it is easy to see that $u[X/p] \approx_{n-1} v[Y/p]$. $\Box$

C: $\mathtt{R^n}$ is necessary for $\mathsf{K5}\pi+$

In this section, we will show that in general, the $\mathtt{R^n}$ axiom is not dispensible. We show that $\mathsf{K}_\Pi\mathtt{5BcAt^3}$ is not an axiomatization of $\mathtt{K5}\pi+$.

First notice that for all n, $\exists p(\Box p \wedge \forall q(\Box q \to \Box^n(p \to q)))$ is valid over all Kripke frames since for arbitrary $w \in W$, $R[w]$ is the witness for it.

We consider the following general frame that falsifies it: $\mathbf{F} = (W, R, B)$ where

$W = \mathbb{N} \cup \{w\}$. Where $w \notin \mathbb{N}$;

$R = \mathbb{N}^2 \cup \{(w, 2n) \mid n \in \mathbb{N}\}$; i.e. R is total in $\mathbb{N}$ and w sees all even numbers;

$B = \{N \subseteq W \mid N$ is finite or cofinite with respect to $W\}$.

Proposition 6.1 $\mathbf{F}$ *is a quantifiable frame,* $\mathbf{F} \vDash \mathrm{K}_\Pi\mathtt{5BcAt^3}$ *but not* $\exists p(\Box p \wedge \forall q(\Box q \to \Box^3(p \to q)))$.

Proof. First let us check that $\mathbf{F}$ is a quantifiable frame. First we show that for the generated subframe of arbitrary $n \in \mathbb{N}$, $\mathbf{F}_n = (\mathbb{N}, \mathbb{N}^2, B_n)$ is a quantifiable frame, where $B_n = \{N \cap \mathbb{N} \mid N \in B\} = \{N' \subseteq \mathbb{N} \mid N'$ is finite or cofinite with respect to $\mathbb{N}\}$. That's because B_n is discrete and clearly all duplicate classes in $\mathbf{F}_n = (\mathbb{N}, \mathbb{N}^2)$ are in B_n, hence by Theorem 4.7, for all assignment $v : \mathsf{Fv}(\varphi) \to B_n$, $[\![\varphi]\!]^{B_n}(v) = [\![\varphi]\!]^{\wp(\mathbb{N})}(v)$. By Lemma 4.6, $[\![\varphi]\!]^{\wp(\mathbb{N})}(v) = [\![\zeta]\!](v)$ for some $\zeta \in \langle \mathsf{Fv}(\varphi)\rangle$. B_n is a field of sets and hence closed under Boolean operations, thus $[\![\zeta]\!](v) \in B_n$. Therefore, $[\![\varphi]\!]^{\wp(\mathbb{N})}(v)$ and hence

$[\![\varphi]\!]^{B_n}(v)$ are in B_n. Therefore, for arbitarary assignment $v : P \to B_n$ and arbitrary φ, $[\![\varphi]\!]^{B_n}(v) \in B_n$. And hence $\mathbf{F}_n$ is a quantifiable frame.

Then, for arbitrary assignment $v : \mathsf{Fv}(\varphi) \to B$, $[\![\varphi]\!]^B(v) \cap \mathbb{N}$ is the interpretation of φ on the subframe $\mathbb{F}_n = (\mathbb{N}, \mathbb{N}^2, B_n)$ under interpretation $v' : p \mapsto v(p) \cap \mathbb{N}$, and hence is in B_n. Consequently, $[\![\varphi]\!]^B(v)$, which at most has one element more than $[\![\varphi]\!]^B(v) \cap \mathbb{N}$, is either finite or cofinite with respect to W, therefore $[\![\varphi]\!]^B(v)$ is in B. Thus, $\mathbf{F}$ is a quantifiable frame.

It is easy to check that $\mathbf{F} \vDash \mathrm{K}_\Pi\mathtt{5BcAt}^3$; $\mathtt{At}^3$ is valid because all singletons are in B. However, since the set of evens is not in the propositional domain, $\mathbf{F}, w \nvDash \exists p(\Box p \wedge \forall q(\Box q \to \Box^3(p \to q)))$: For arbitrary $X \in B$, if $\mathbf{F}, v[X/p], w \vDash \Box p$, then X must be cofinite and contains all even numbers. Delete an odd number from it and assign the subset we get to q. Then the assignment would still satisfy $\Box q$ for the interpretation of q still contains all the even numbers; and it would satisfy $\Diamond^3(p \wedge \neg q)$ for on the odd number that was deleted from X, $p \wedge \neg q$ is true. Hence $\mathbf{F}, v[X/p], w \vDash \Box p \to \exists q(\Box q \wedge \Diamond^3(p \wedge \neg q))$. This completes the proof. □

Hence we conclude that:

Corollary 6.2 $\mathsf{K}_\Pi\mathtt{5BcAt}^3 \neq \mathtt{K5}\pi+$.

D: Sahlqvist condition is necessary

We show that in our result Theorem 5.2 the Sahlqvist condition cannot be dropped completely. Without it, Kripke incompleteness for the quantifier-free fragment can lead to incompleteness for the full propositionally quantified language even with $\mathtt{At}^n$ and $\mathtt{R}^n$. We know that $\mathtt{TMEQ}$ defines the class of Kripke frames with the identity accessibility relation and is Kripke incomplete [27]. The axioms are: $\mathtt{T} : \Box p \to p$, $\mathtt{M} : \Box\Diamond p \to \Diamond\Box p$, $\mathtt{E} : \Diamond(\Diamond p \wedge \Box q) \to \Box(\Diamond p \vee \Box q)$, $\mathtt{Q} : (\Diamond p \wedge \Box(p \to \Box p)) \to p$. Now we show the following:

Theorem 6.3 $\mathsf{K}_\Pi\mathtt{TMEQBcAt}^n\mathtt{R}^n \neq \mathtt{TMEQ}\pi+$.

Consider the veiled recession frame: $(\mathbb{Z}, R, B)$ where nRm iff $m \geqslant n-1$. B is the collection of sets that are eventually settled after some $n \in \mathbb{Z}$, that is, $B = \{X \subseteq \mathbb{Z} \mid X \supseteq R[n] \text{ or } \mathbb{Z} - X \supseteq R[n] \text{ for some } n \in \mathbb{Z}\}$. For a valuation $v : \mathsf{Prop} \to B$, we say that $p \in P$ is eventually settled as true by v after n, if $m \in v(p)$ for all $m \geq n$, and $p \in P$ is eventually settled as false by v after n, if $m \notin v(p)$ for all $m \geq n$.

Definition 6.4 A *general model* $\mathcal{M} = (W, R, B, v)$ is a tuple where (W, R, B) is a general frame and v is a valuation from Prop to B. Given a general moodel $\mathcal{M} = (W, R, B, v)$ and $x \in W$, its *x-generated submodel* of degree n is $\mathcal{M}^n_x = (W^n_x, R|_{W^n_x}, B^n_x, v^n_x)$ where $W^n_x = \bigcup_{i \leq n} R^n[x]$, $B^n_x = \{X \cap W^n_x \mid X \in B\}$ and $v^n_x(p) = v(p) \cap W^n_x$. An x-generated subframe of degree n $\mathbf{F}^b_x$ of $\mathbf{F}$ is defined similarly by disregarding the valuation component.

For any $\varphi \in \mathcal{L}$, the modal depth $md(\varphi)$ of φ is defined as usual with propositional quantifiers ignored.

The following lemma has been shown for example in [6].

Lemma 6.5 *For arbitrary propositionally quantified modal logic formula φ and natural number n not smaller than the modal depth of φ, for arbitrary general frame $\mathbf{F}$ and $x \in W$, $x \in [\![\varphi]\!]^{\mathbf{F}}(v)$ iff $x \in [\![\varphi]\!]^{\mathbf{F}^n_x}(v^n_x)$.*

Lemma 6.6 $(\mathbb{Z}, R, B)$ *is a quantifiable frame that validates* $\mathtt{T}, \mathtt{M}, \mathtt{E}, \mathtt{Q}, \mathtt{At^n}$ *and* $\mathtt{R^n}$ *for arbitrary $n \geq 1$.*

Proof. It is easy to verify that the frame validates $\mathtt{T}$. $\mathtt{M}$ is valid since every set in B is eventually settled. To verify $\mathtt{E}$, notice if a number n satisfies $\Diamond p \wedge \Box q$, then all numbers smaller than n would satisfy $\Diamond p$ and numbers greater would satisfy $\Box q$. To verify $\mathtt{Q}$, notice for arbitrary nRm there is a finite path from n back to m, hence if a world satisfies $\Diamond p \wedge \Box(p \to \Box p)$, then it sees a world with p and p can be forced back to itself since all worlds along the path has $p \to \Box p$. To see that it validates $\mathtt{At^n}$ and $\mathtt{R^n}$, notice all singletons are in B and all $R[X]$ are in B for arbitrary $X \subseteq \mathbb{Z}$.

Now we verify that it is a quantifiable frame: We omit the easy Boolean cases. For model case: if $[\![\varphi]\!]^{\mathbf{F}}(v)$ is settled as true at n, then $[\![\Diamond\varphi]\!]^{\mathbf{F}}(v) = \mathbb{Z}$; if $[\![\varphi]\!]^{\mathbf{F}}(v)$ is settled as false at n, then $[\![\Diamond\varphi]\!]^{\mathbf{F}}(v)$ is settled as false at $n+1$.

For the quantifier case, let l be a integer s.t. all $q \in \mathsf{Fv}(\exists p\varphi)$ is settled by v after l. Then one of the following is the case:

- For any $X \in B$ and any $n \geqslant l + md(\varphi)$, $n \notin [\![\varphi]\!]^{\mathbf{F}}(v[X/p])$.
- There is $X \in B$ and $n \in [\![\varphi]\!]^{\mathbf{F}}(v[X/p])$ s.t. $n \geqslant l + md(\varphi)$.

If the former, then $[\![\exists p\varphi]\!]^{\mathbf{F}}(v)$ is settled as false after $l + md(\varphi)$. If the latter, then for arbitrary $m \in \mathbb{N}$, consider the n-generated submodel of degree $md(\varphi)$ of $(\mathbf{F}, v[X/p])$ and the $(n+m)$-generated submodel of degree $md(\varphi)$ of $(\mathbf{F}, v[X+m/p])$, they are model isomorphic by the map $\pi : x \mapsto x + m$ since all $q \in \mathsf{Fv}(\varphi) \setminus \{p\}$ are settled before $n - md(\varphi)$ (We justifiably neglect the valuation of proposition letters not in $\mathsf{Fv}(\varphi)$). Consequently, by Lemma 6.5, $n \in [\![\varphi]\!]^{\mathbf{F}}(v[X/p])$ iff $n \in [\![\varphi]\!]^{\mathbf{F}^{md(\varphi)}_n}(v[X/p]^{md(\varphi)}_n)$ iff $n + m \in [\![\varphi]\!]^{\mathbf{F}^{md(\varphi)}_{n+m}}(v[X+m/p]^{md(\varphi)}_{n+m})$ iff $n + m \in [\![\varphi]\!]^{\mathbf{F}}(v[X+m/p])$. Since $m \in \mathbb{N}$ is arbitrary, $[\![\exists p\varphi]\!]^{\mathbf{F}}(v)$ is settled as true at n. This finishes the proof.

□

Finally, notice $(\mathbb{Z}, R, B)$ does not validate $p \to \Box p$, hence $p \to \Box p \notin \mathsf{K}_\Pi\mathtt{TMEQBcAt}^n\mathtt{R}^n$. But since $\mathtt{TMEQ}$ defines the Kripke frames with identity accessibility relation, $p \to \Box p$ is valid on this frame class and hence $p \to \Box p \in \mathtt{TMEQ}\pi+$. This finishes the proof.